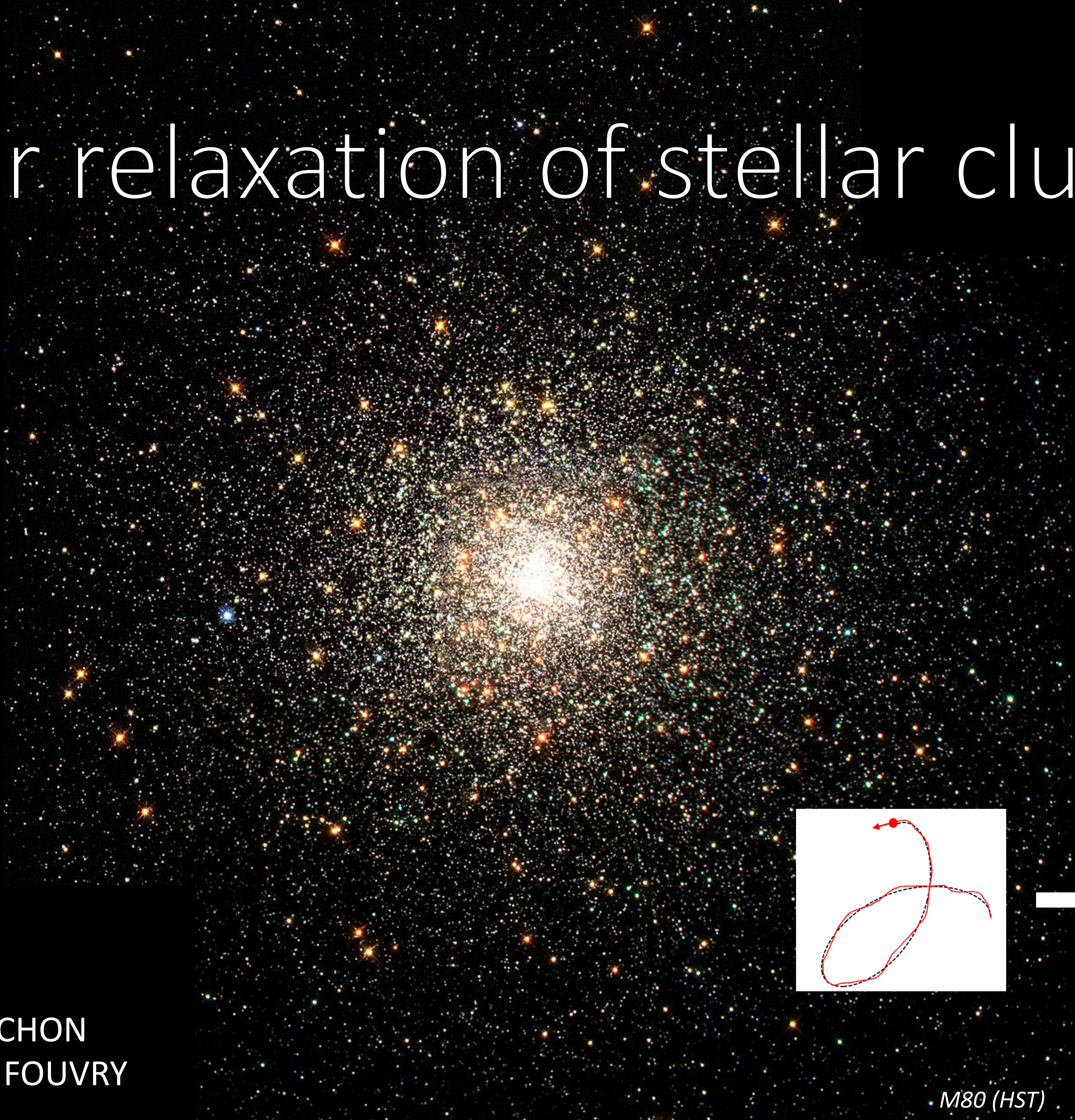
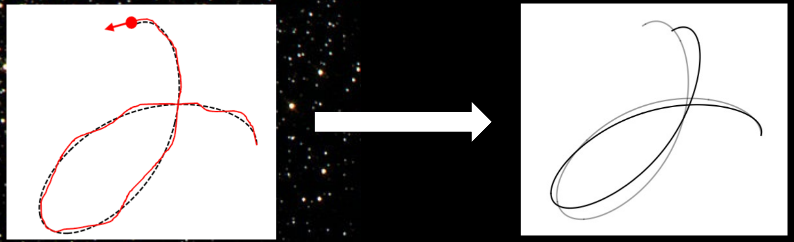
Secular relaxation of stellar clusters



M80 (HST)



Kerwann TEP (IAP)
September 18th, 2023

Supervisors: Christophe PICHON
Jean-Baptiste FOUVRY

Observations

- GAIA, JWST, Euclid
- Statistical description of stellar clusters
- Secular times: good fraction of the age of the Universe

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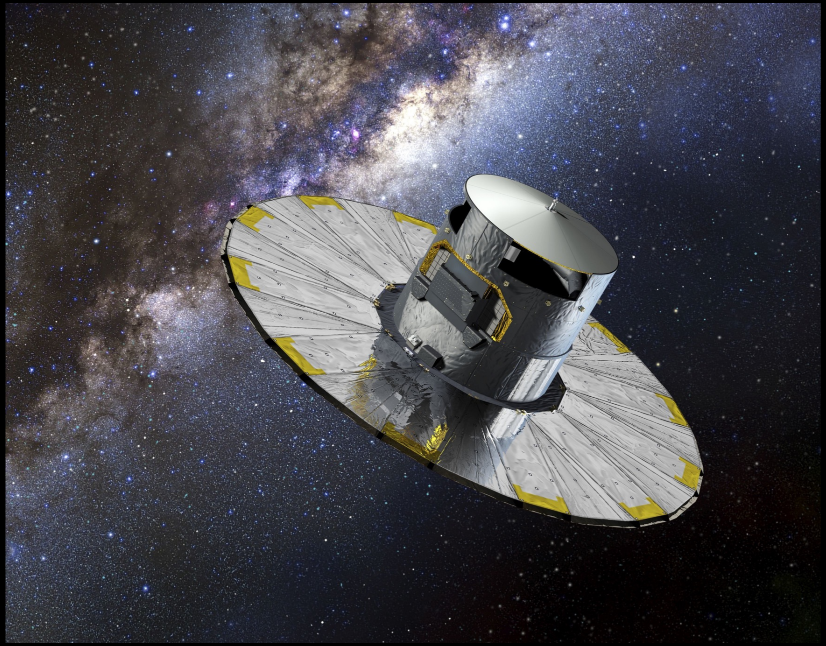
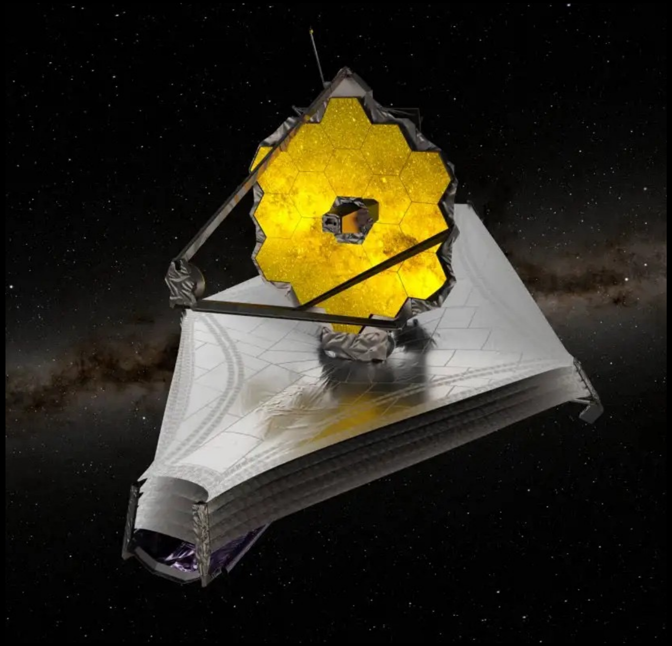
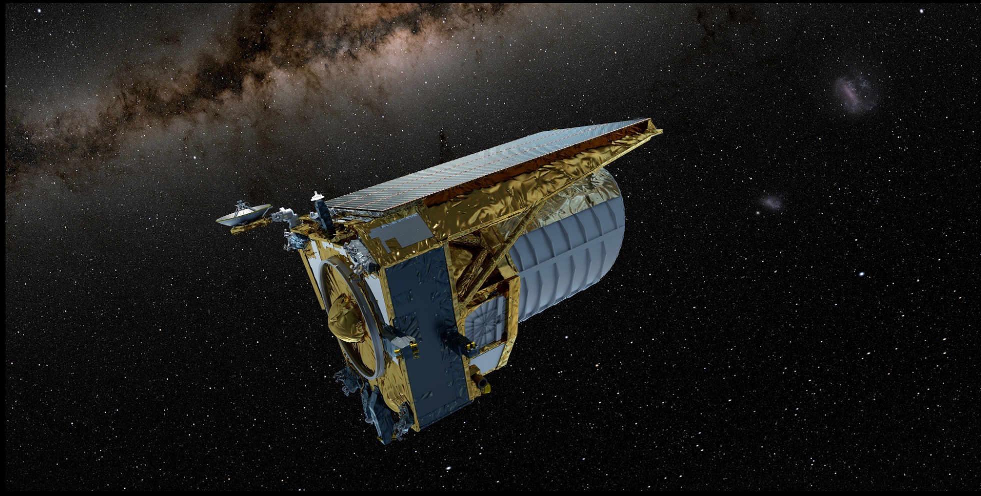
GAIA



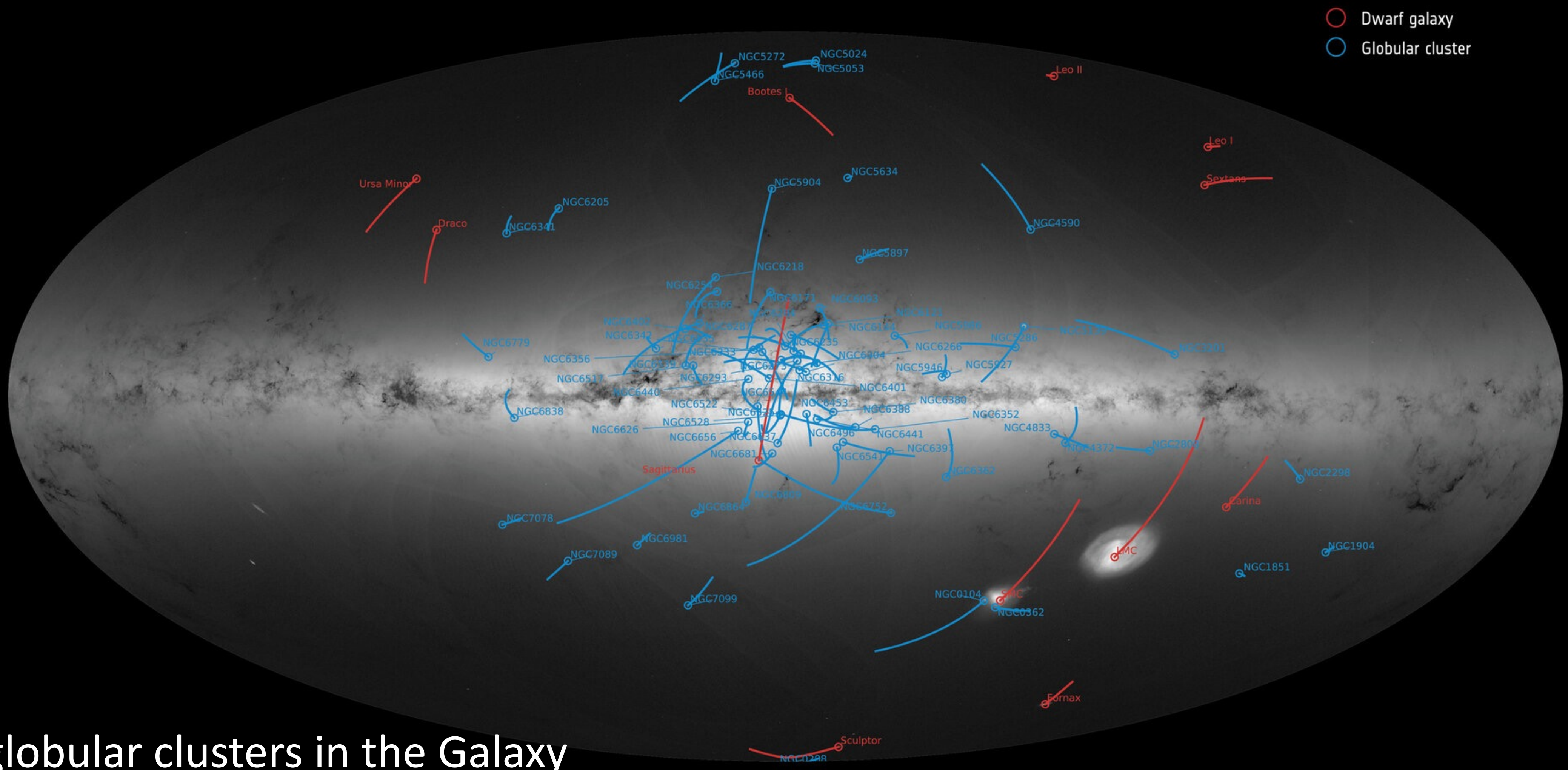
JWST



Euclid



→ GAIA'S GLOBULAR CLUSTERS AND DWARF GALAXIES



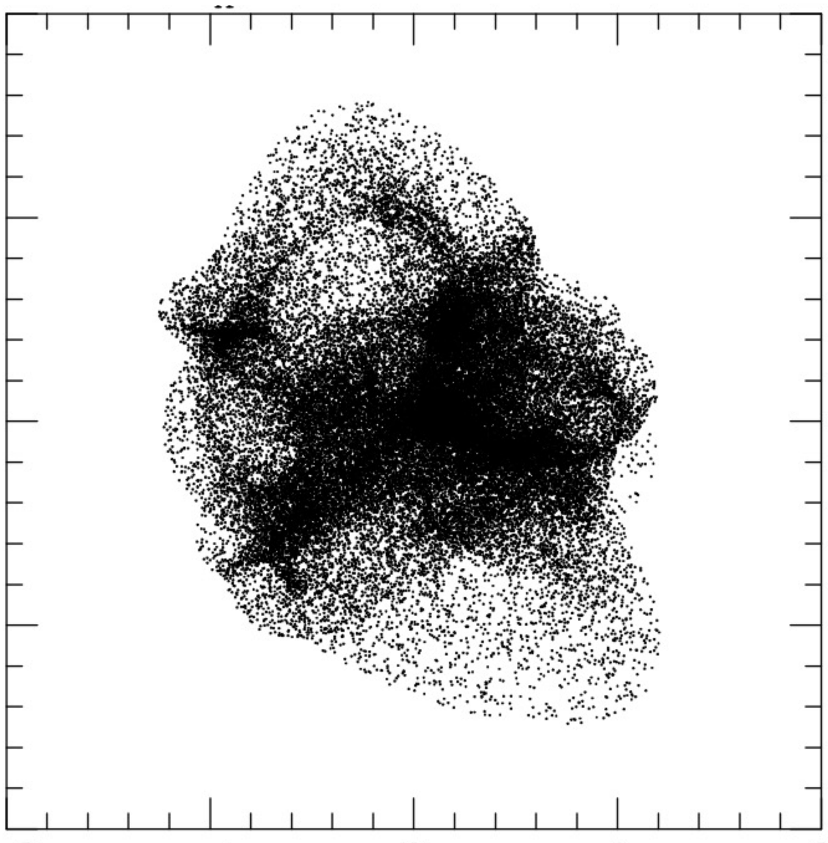
→ 150 globular clusters in the Galaxy

JWST

→ 50,000 sources of near-infrared light

Pandora's Cluster
Credits: NASA, ESA, CSA

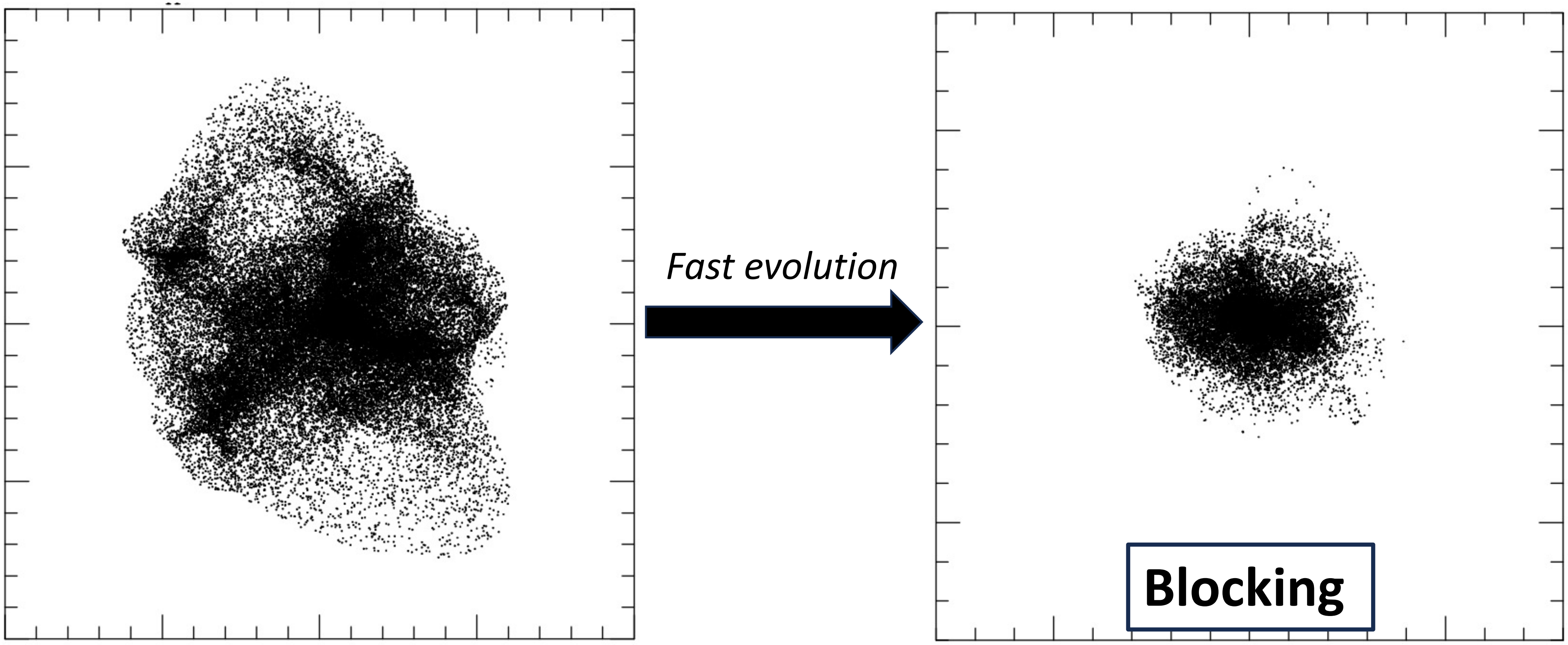
Violent relaxation



Initial conditions

Binney & Tremaine (2008)

Violent relaxation



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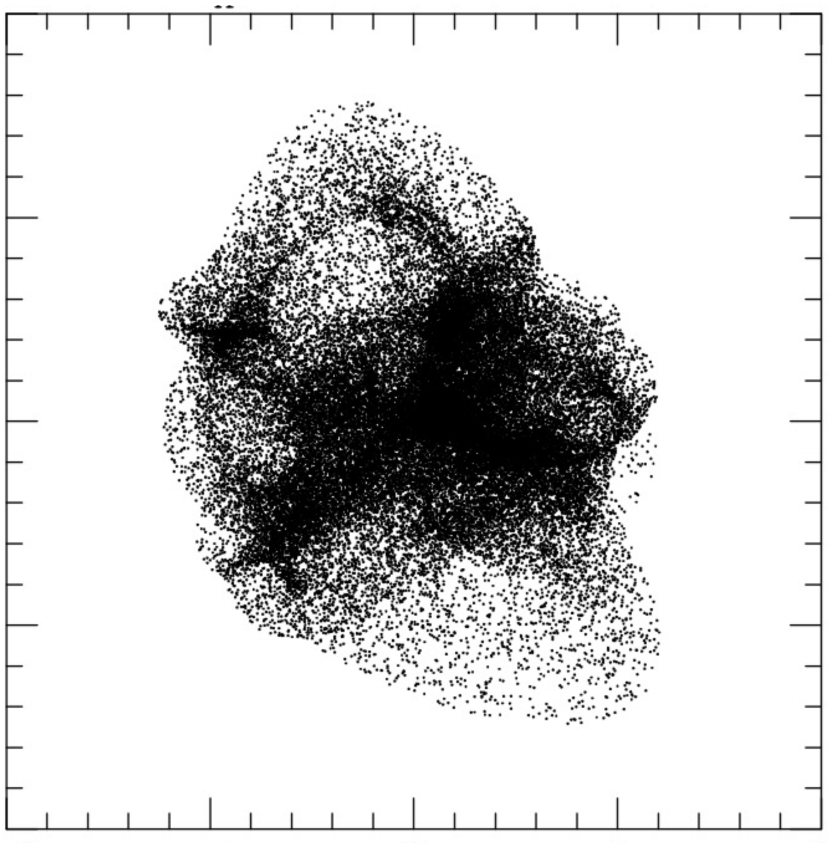
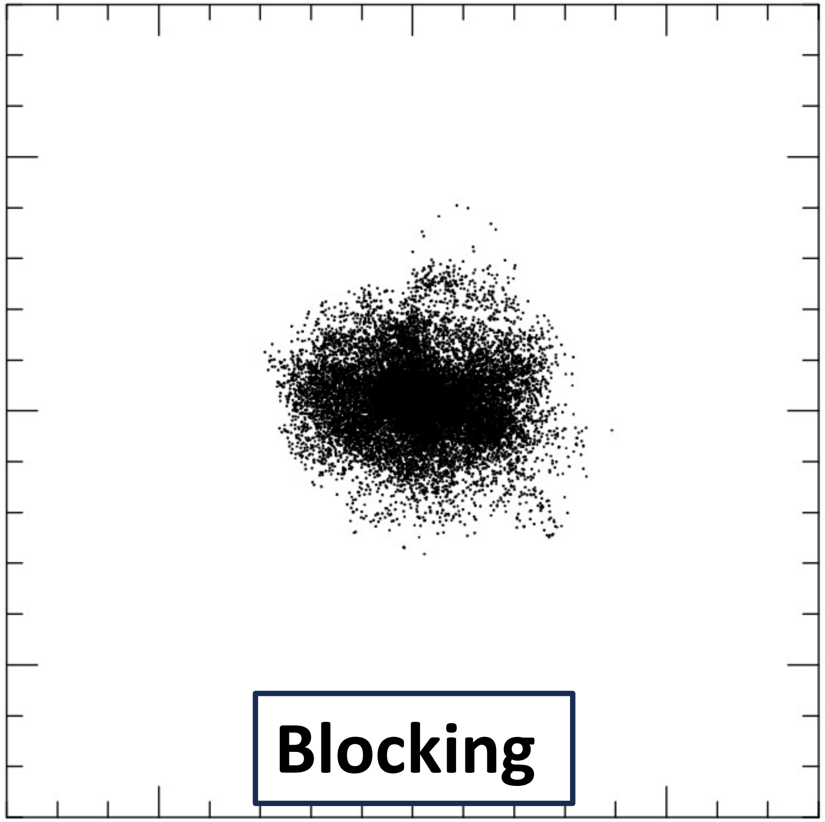


Initial conditions

Fast evolution



→ **Symmetric configuration**

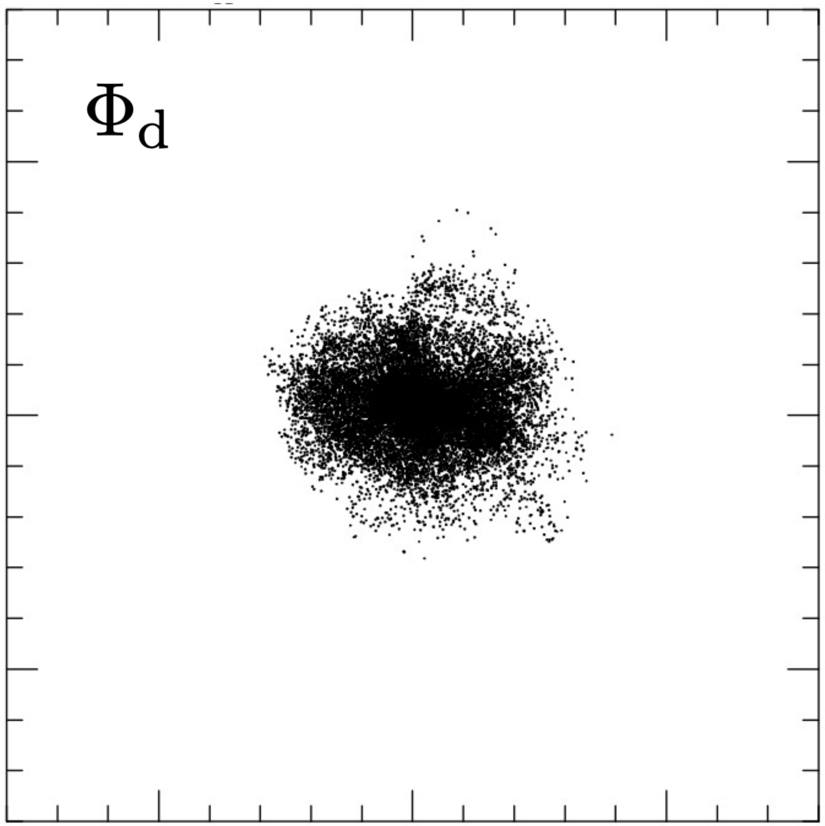
→ **Quasi-stationary state (QSS)**

Binney & Tremaine (2008)

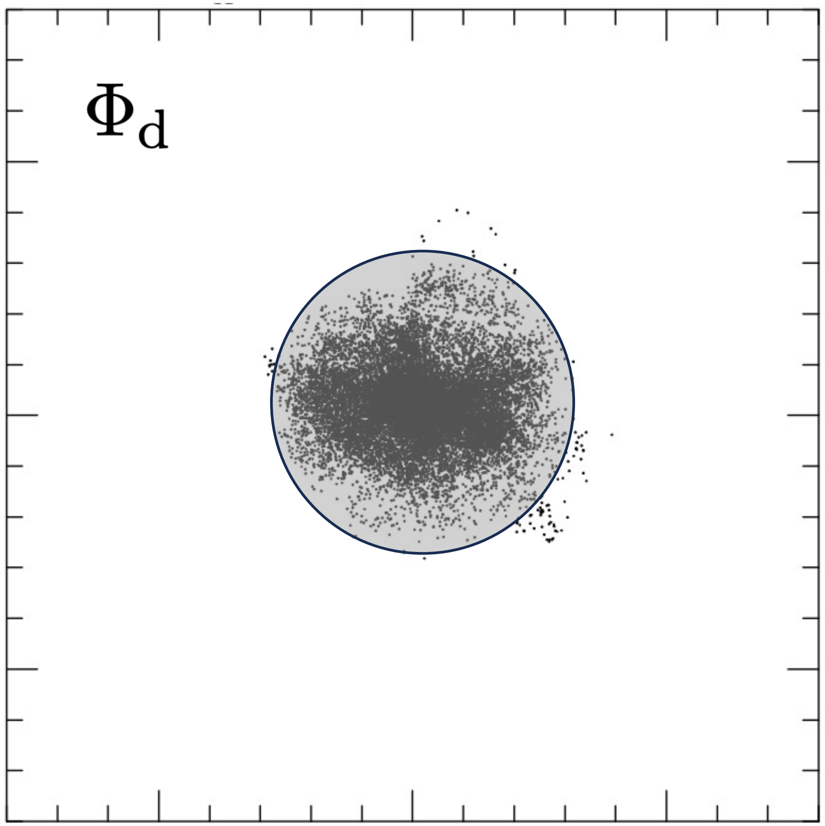
Mean-field limit

Discrete potential

$$\Phi_d = \Phi + \delta\Phi$$



Mean-field limit



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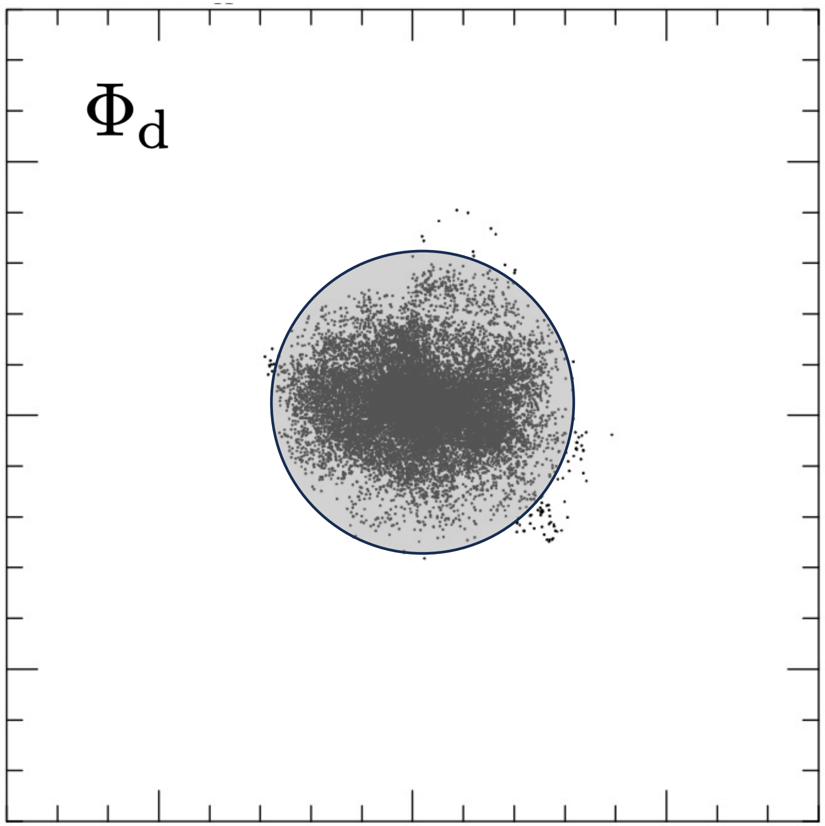
Discrete potential

Mean field

Mean-field limit



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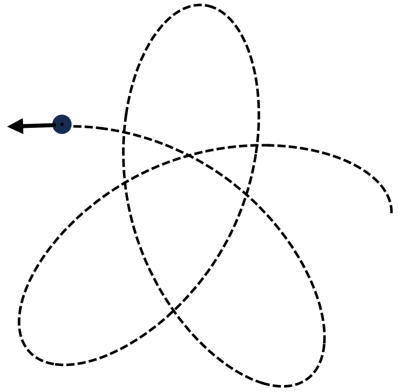
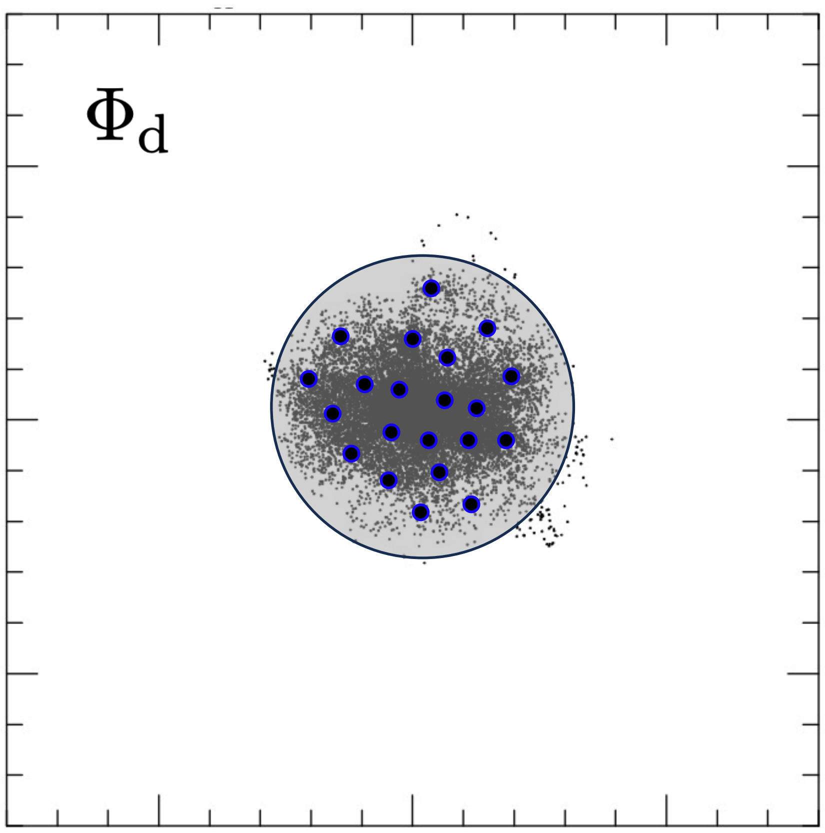


→ Symmetry of QSS

→ Orbit labelling: actions **J**

Fluctuations



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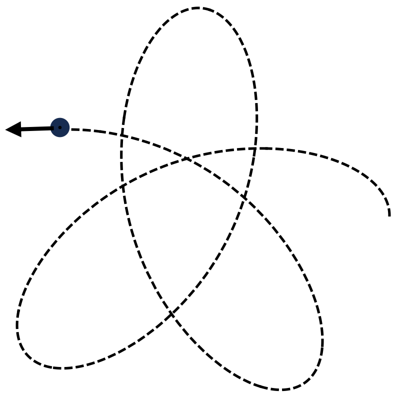

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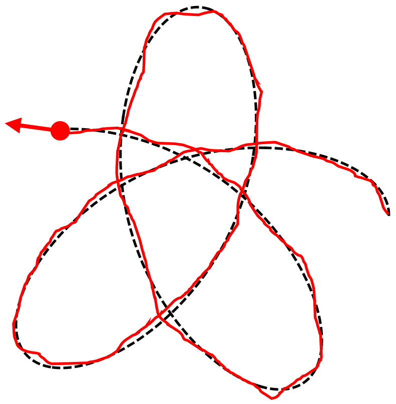


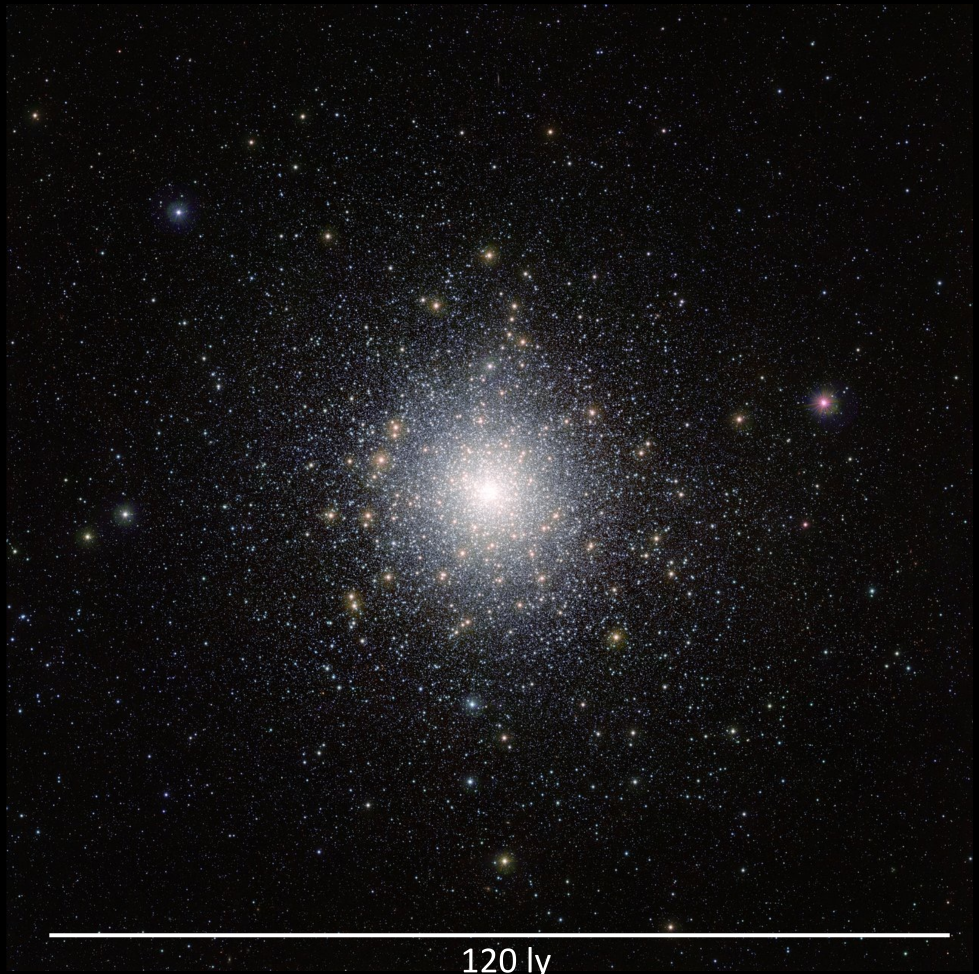
→ Departure from mean-field

→ Distortion of the orbits

→ Slow evolution of QSS

Secular relaxation

N ≃ 500 000



47 Tuc (VISTA)

Credit: *ESO/M.-R. Cioni/VISTA Magellanic Cloud survey.*

Secular relaxation

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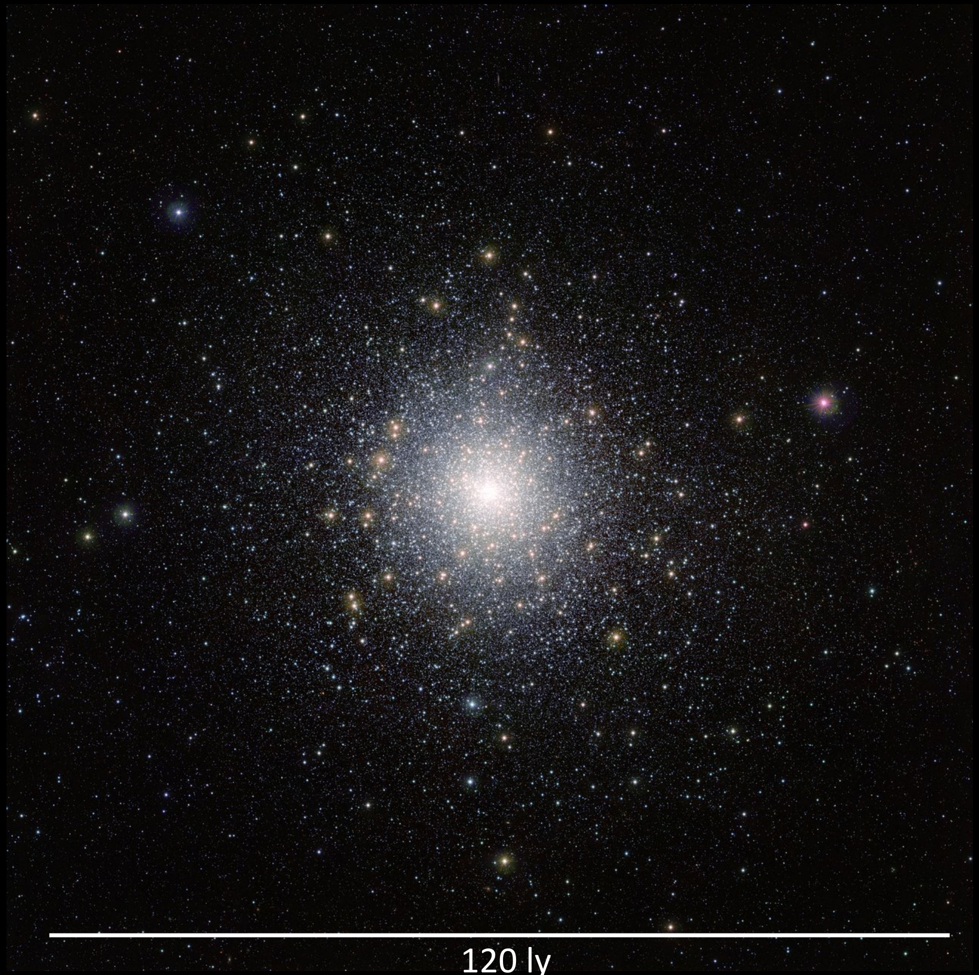

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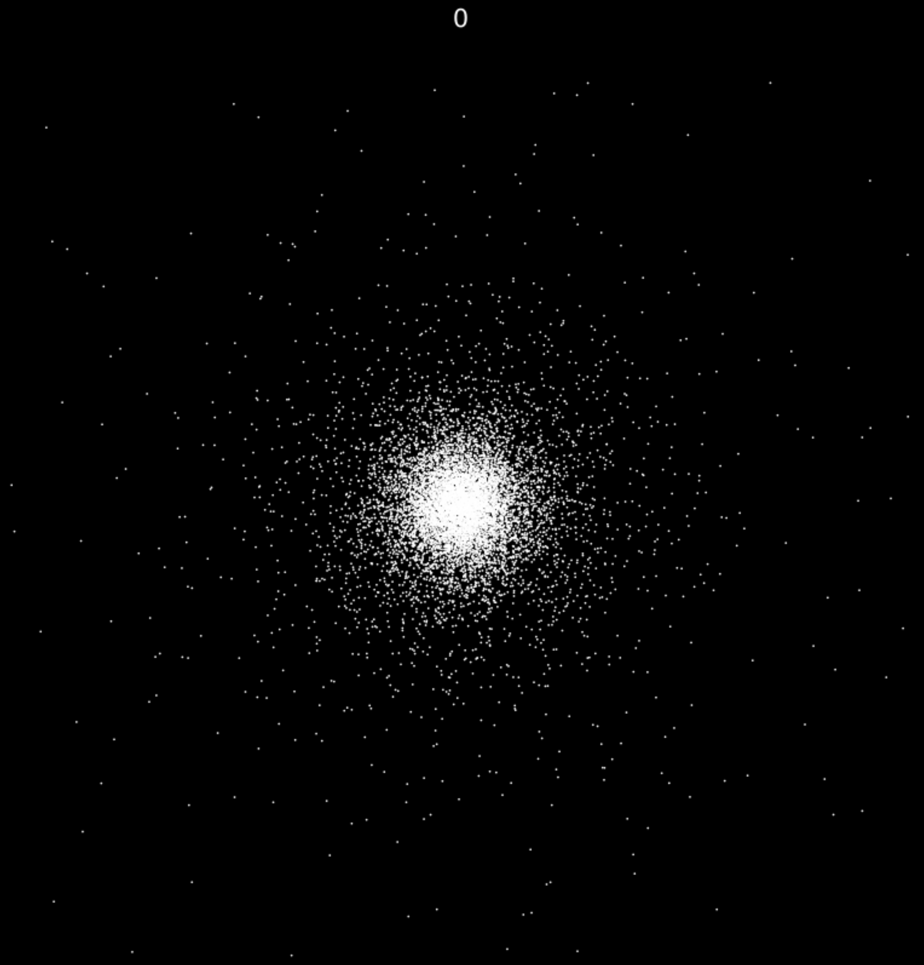

Plummer cluster (N-body)

Credit: *ESO/M.-R. Cioni/VISTA Magellanic Cloud survey.*

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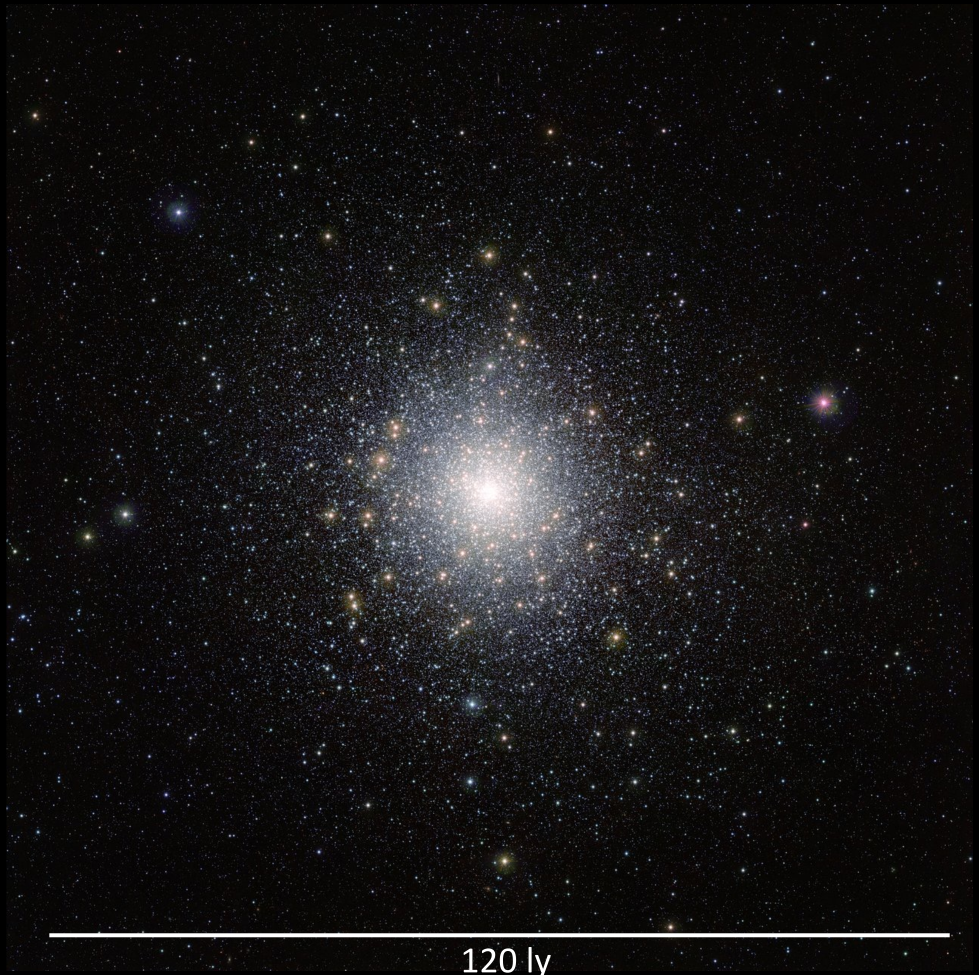

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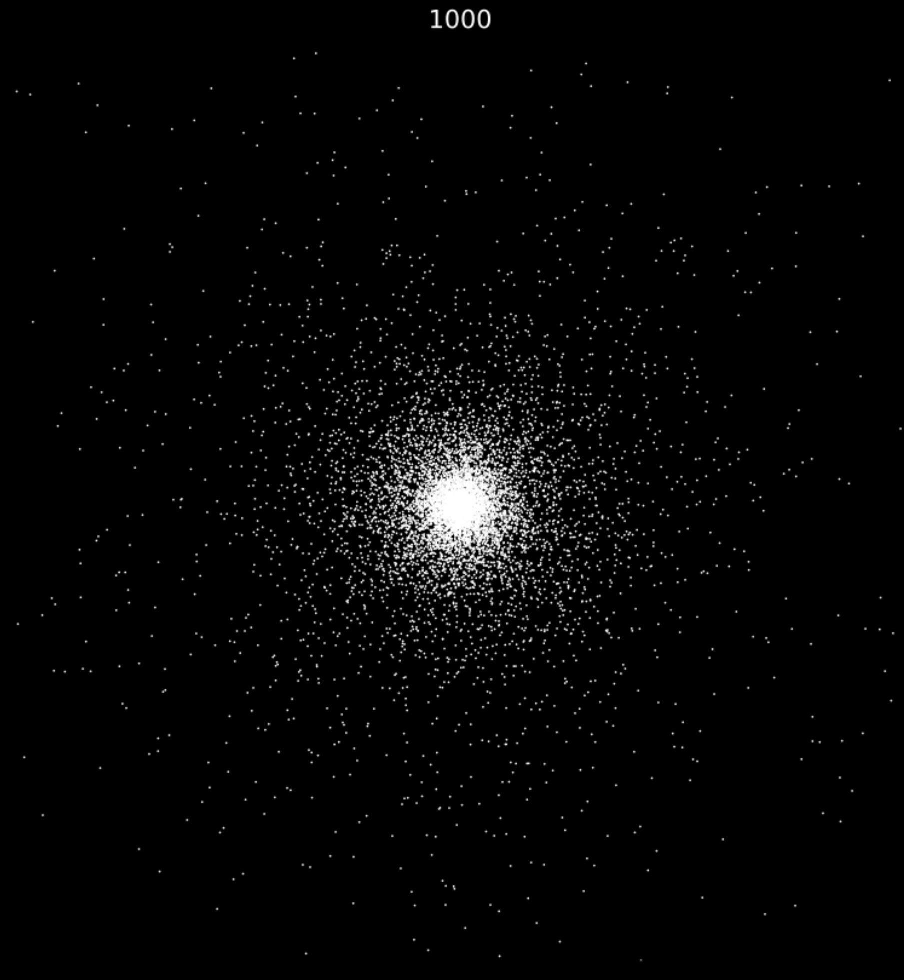

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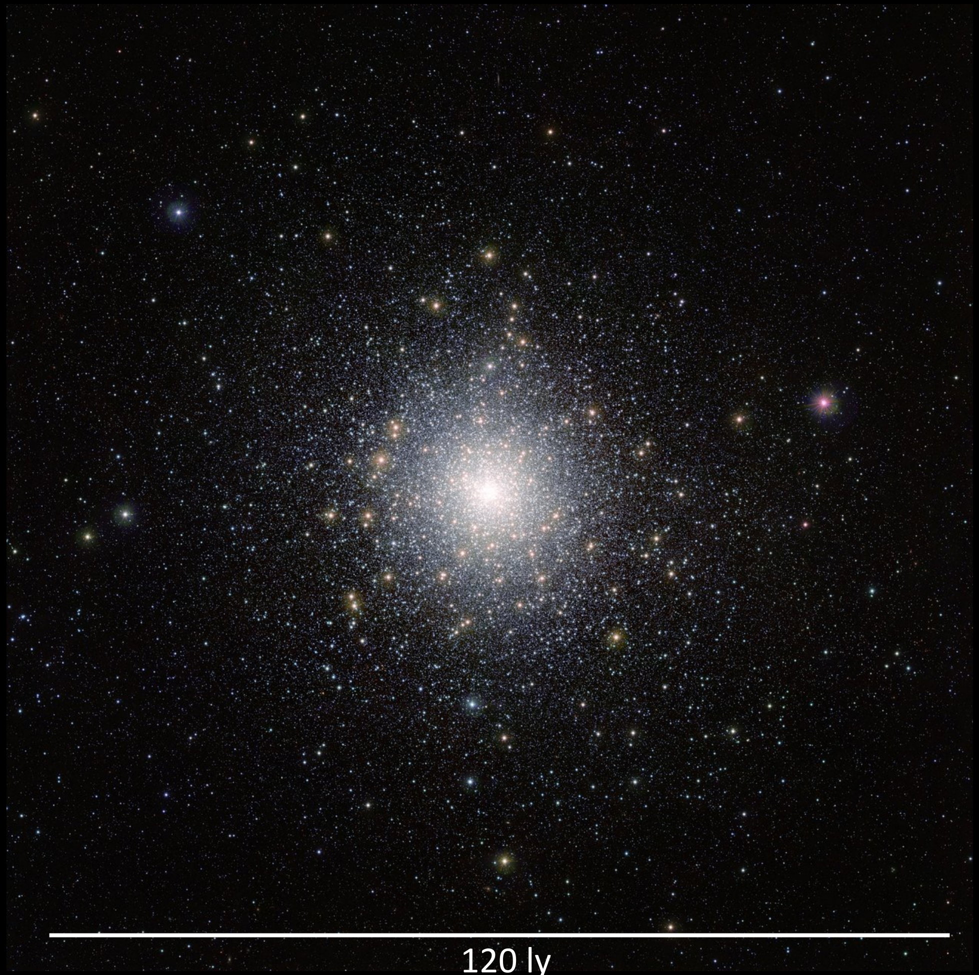

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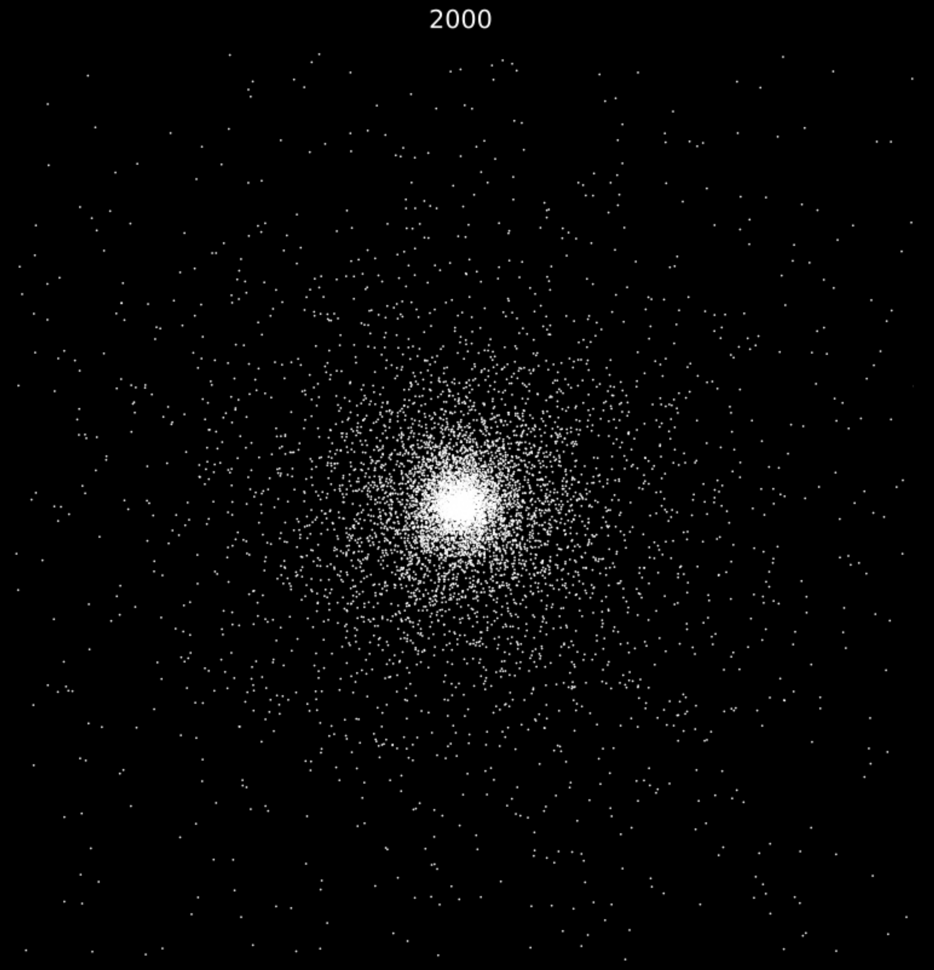

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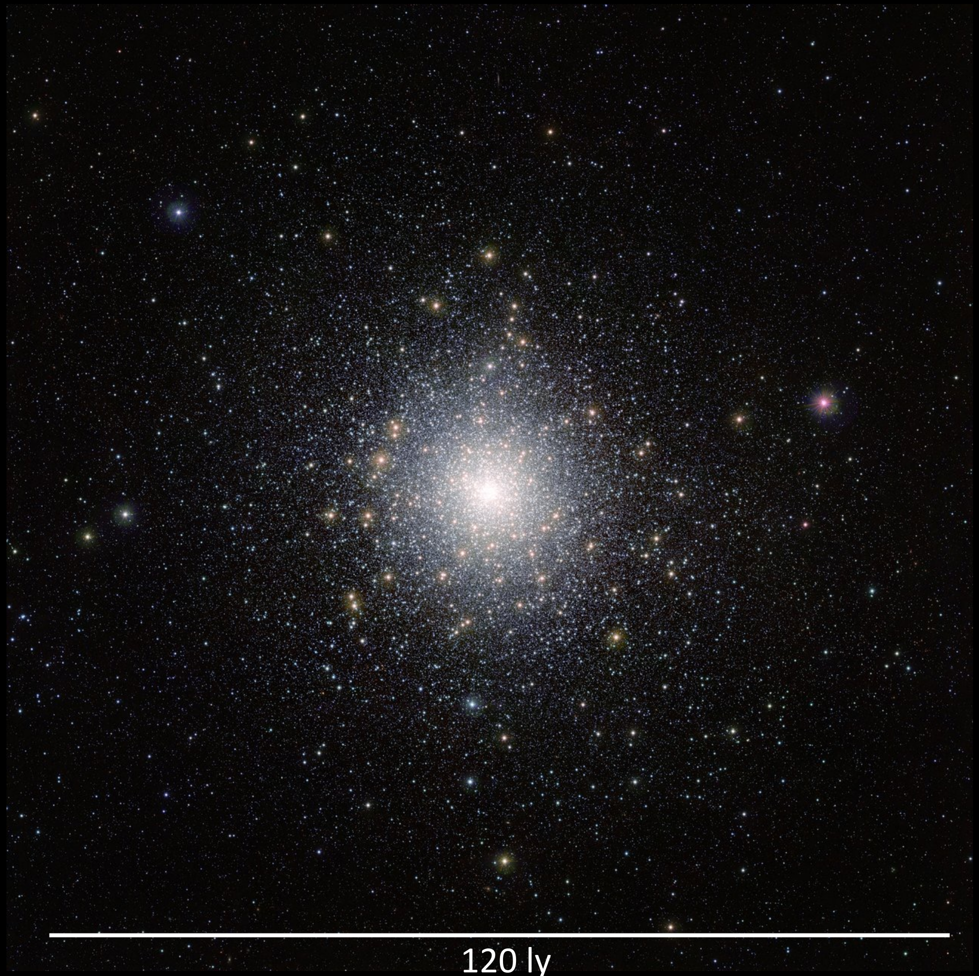

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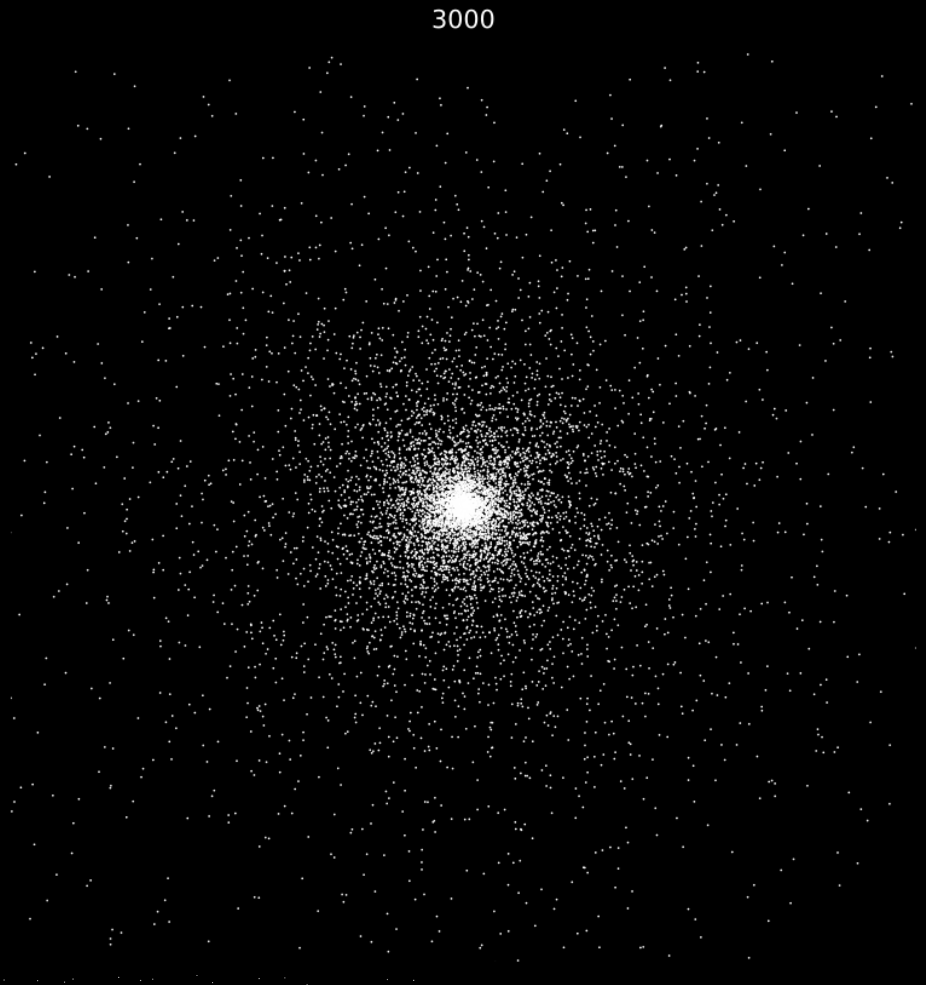
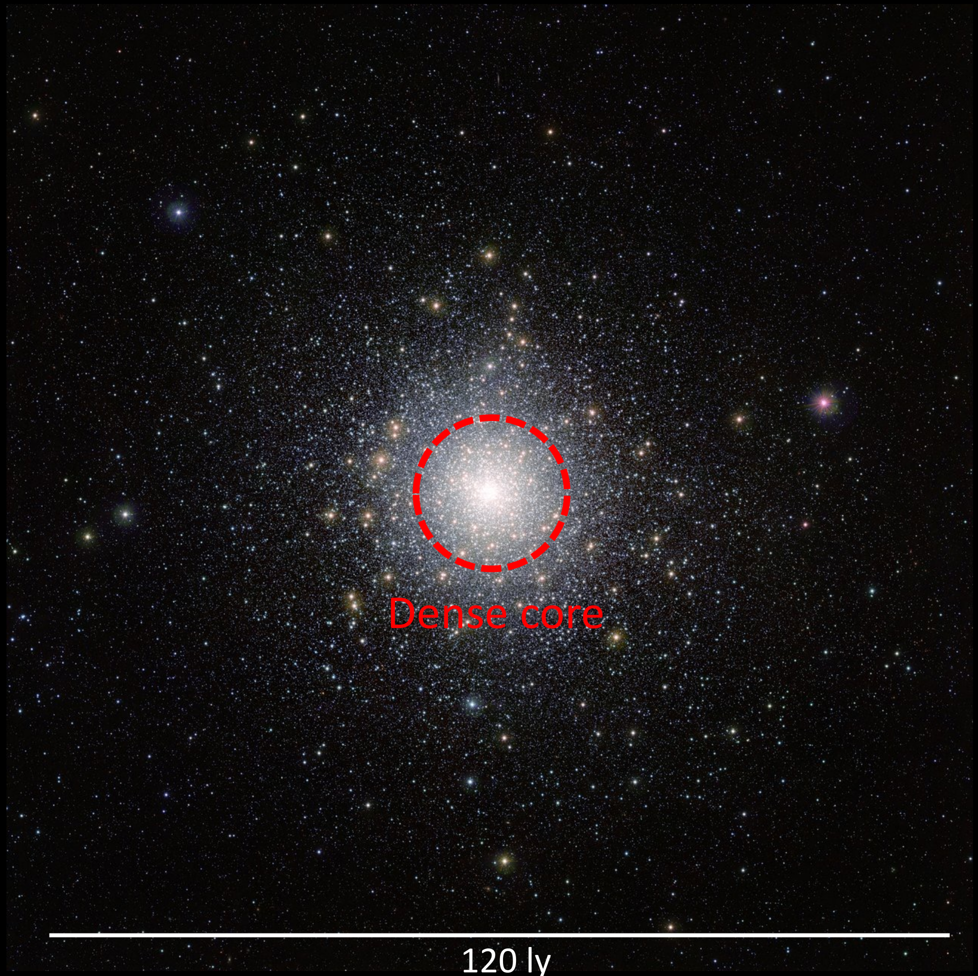


Plummer cluster (N-body)

Credit: *ESO/M.-R. Cioni/VISTA Magellanic Cloud survey.*

Core collapse

N ≃ 500 000



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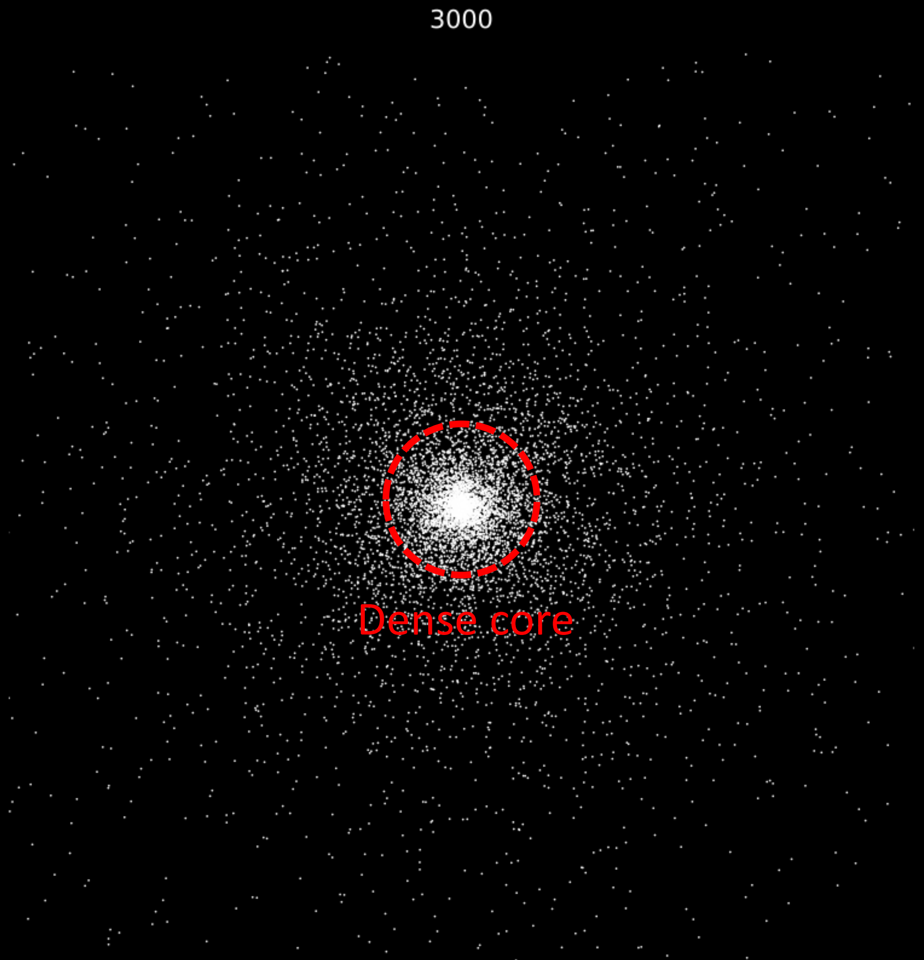

Plummer cluster (N-body)

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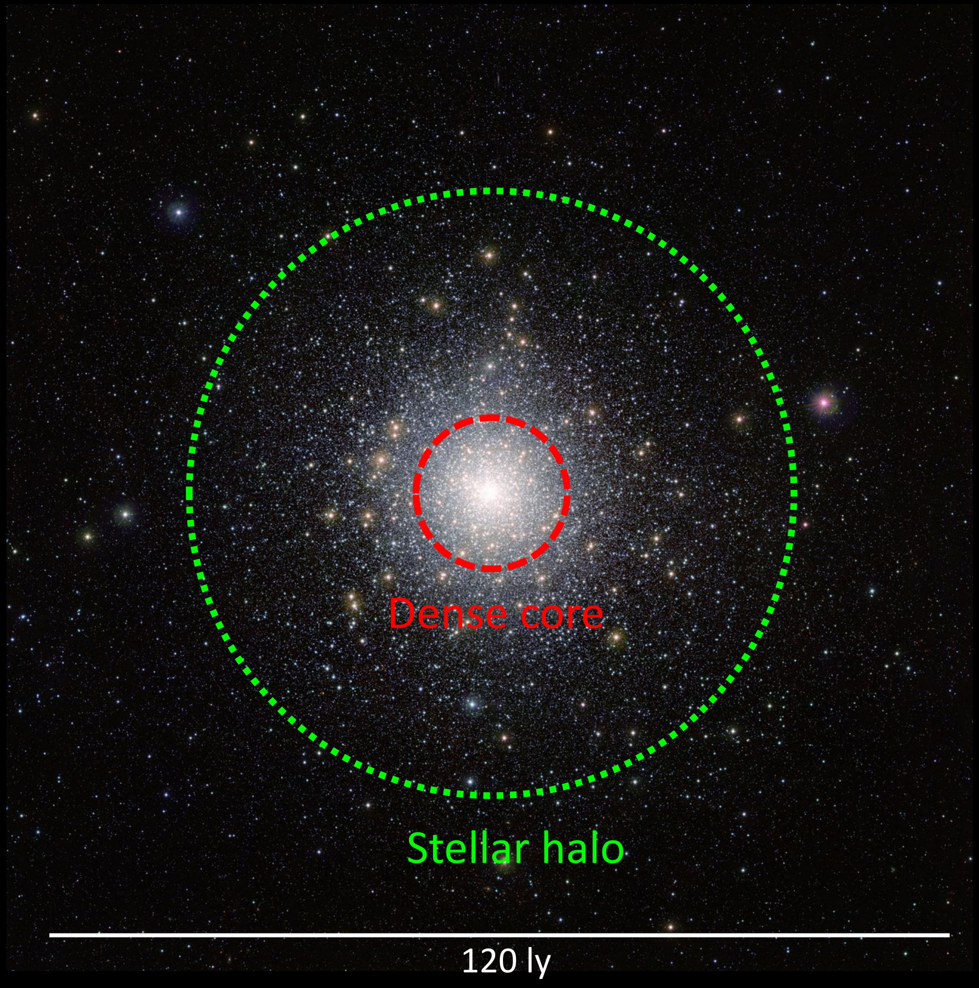

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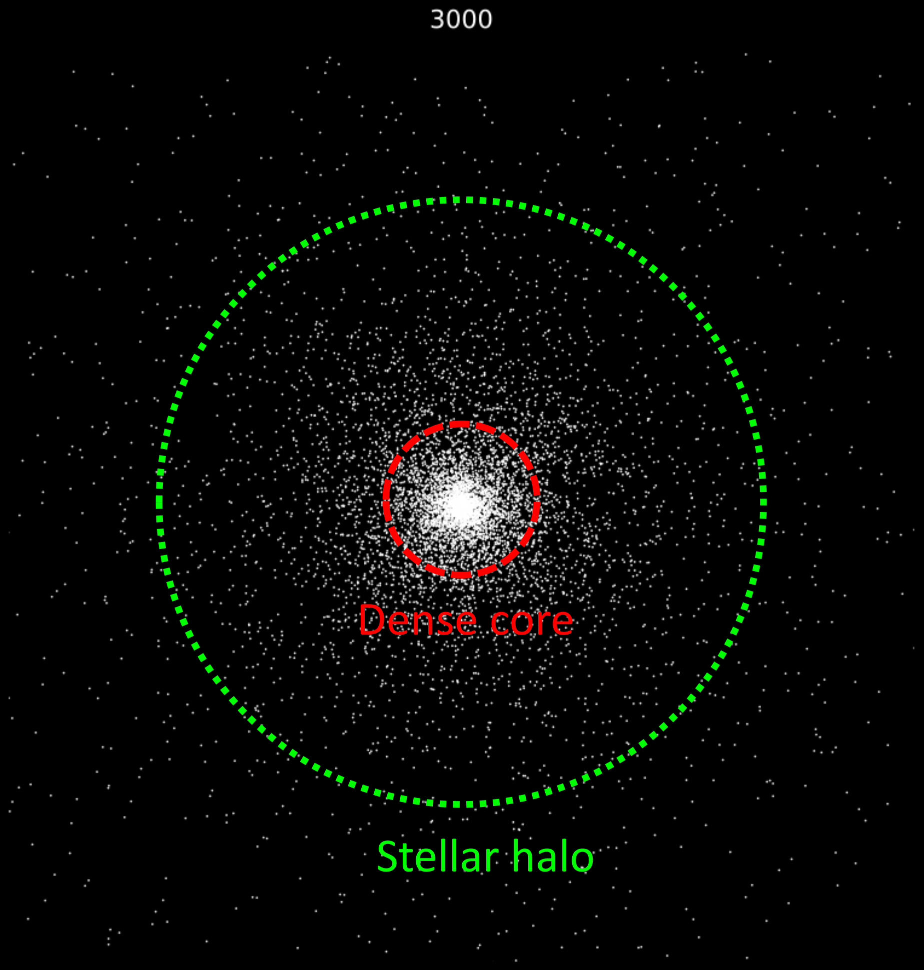


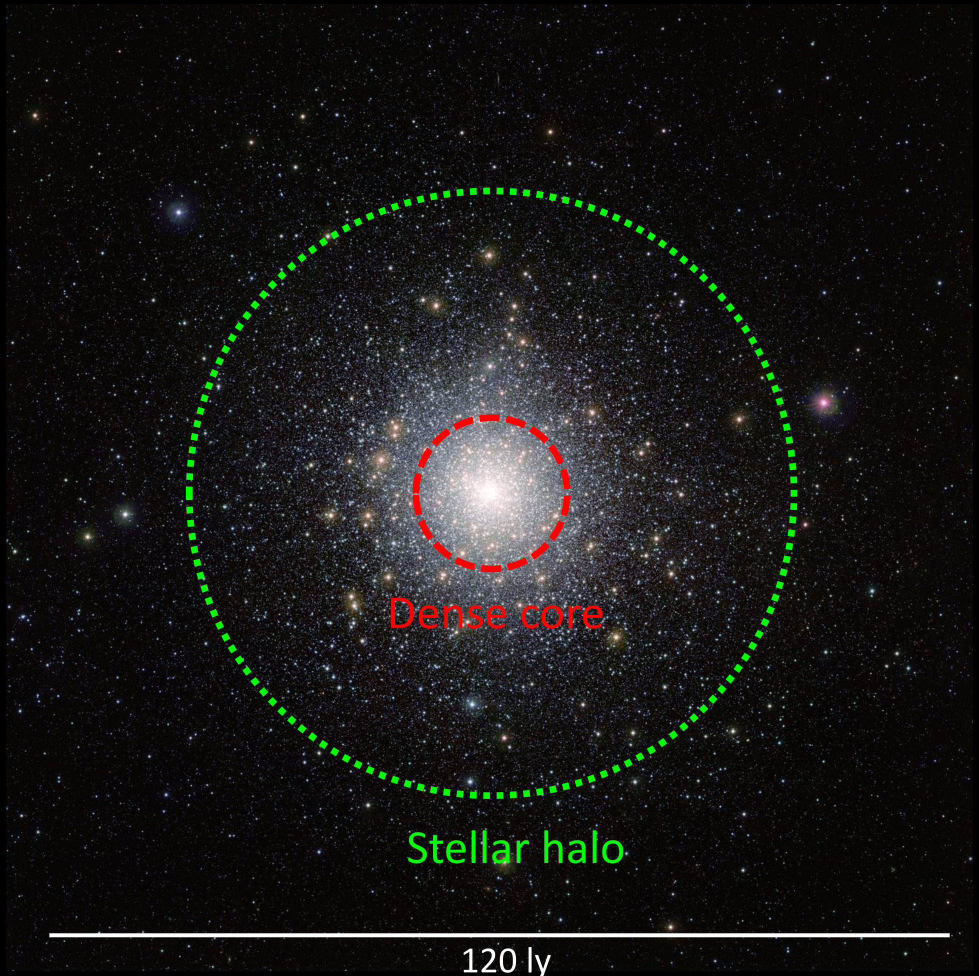
Plummer cluster (N-body)

Credit: *ESO/M.-R. Cioni/VISTA Magellanic Cloud survey.*

Core collapse

→ **What impacts the rate of core collapse ?**

N ≃ 500 000



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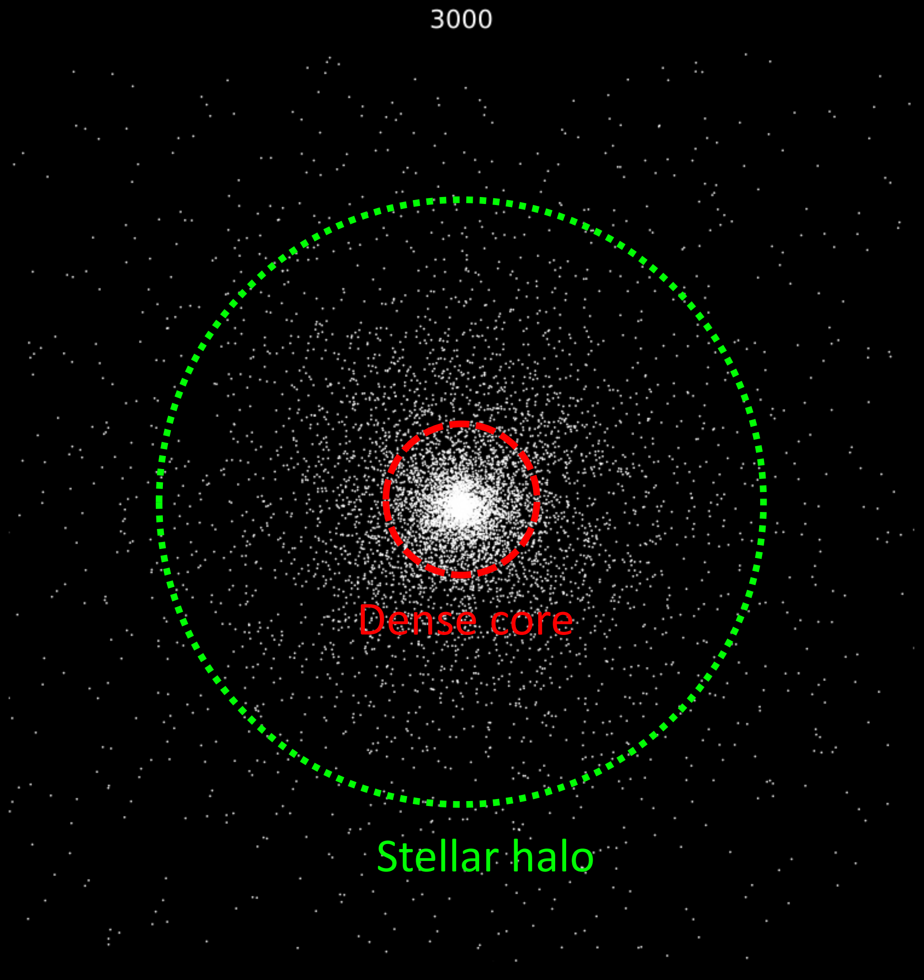

Plummer cluster (N-body)

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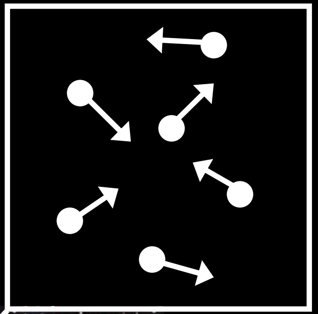
Hot systems

Globular cluster (NGC 1781)



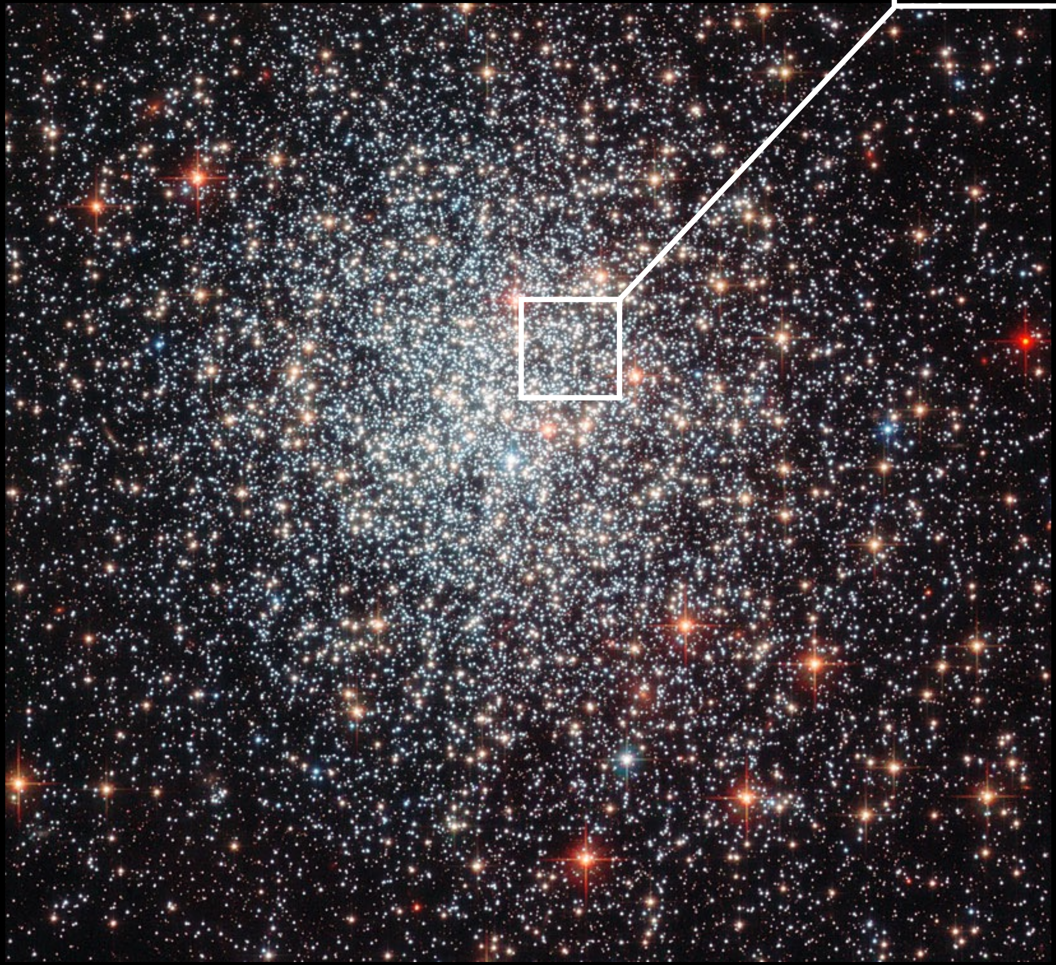
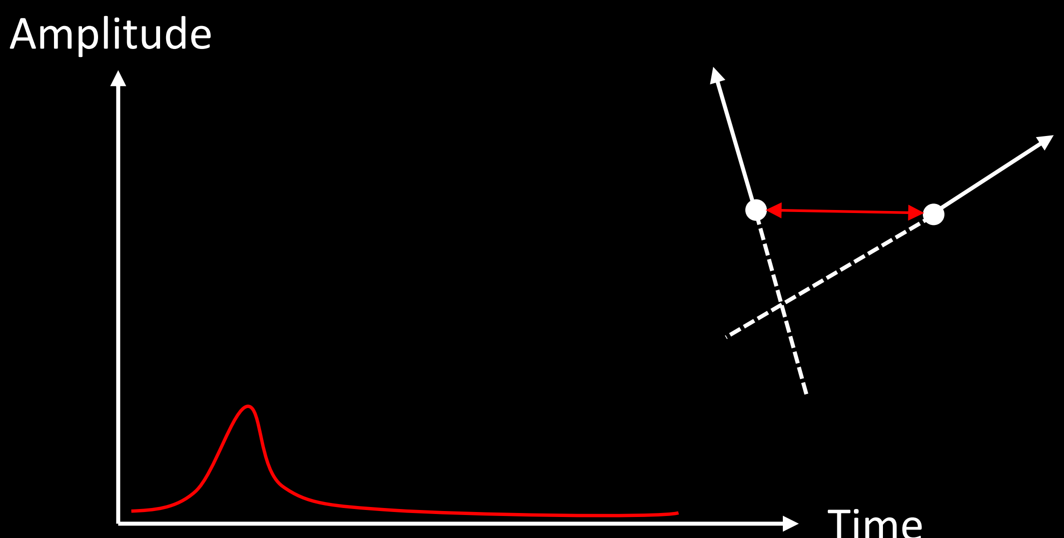


Image credit: ESA/Hubble & NASA HST

Hot systems



Short pairwise interactions

Globular cluster (NGC 1781)

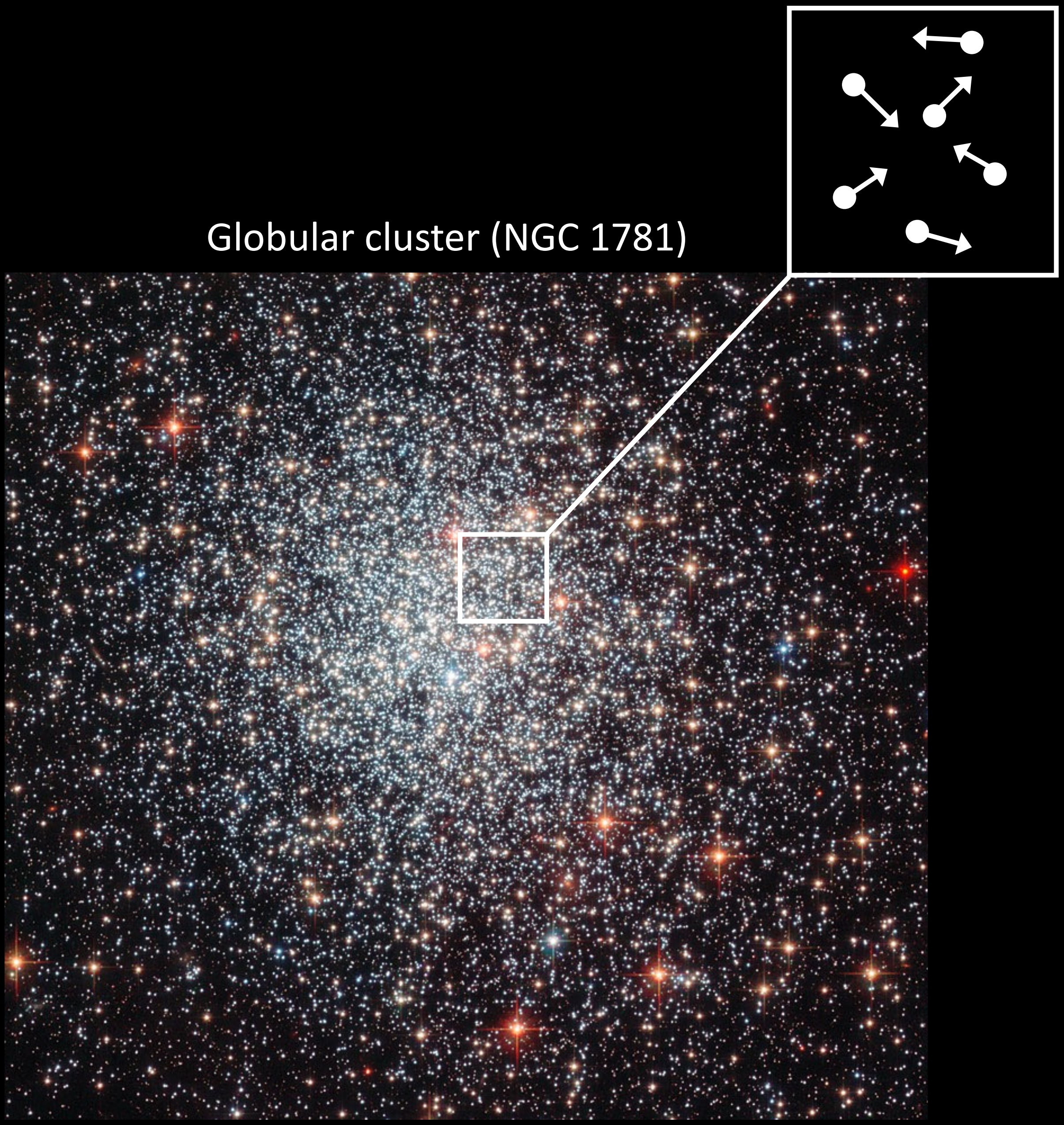
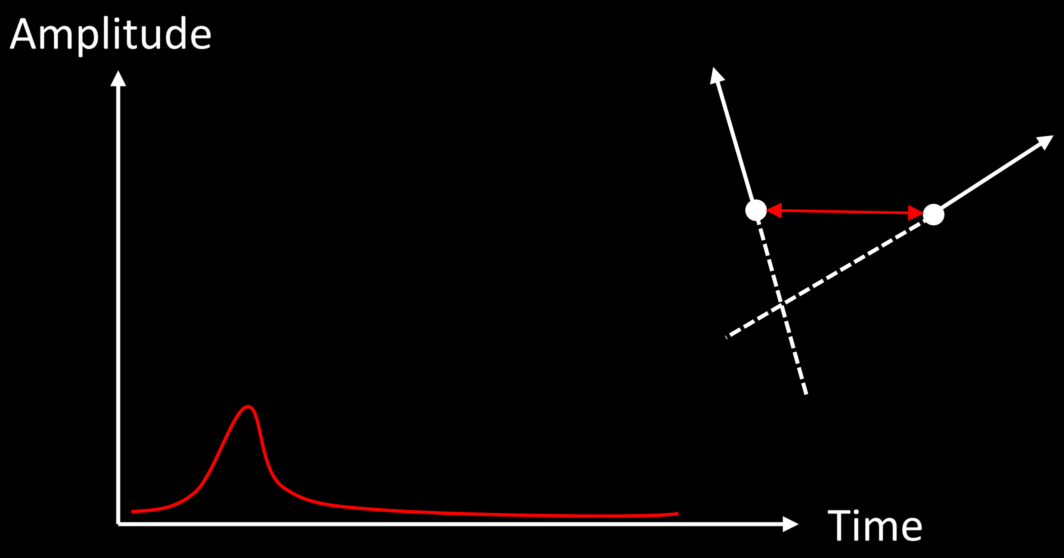
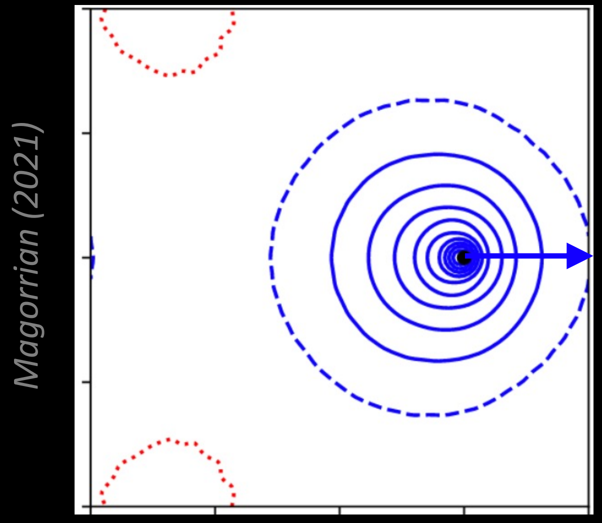


Image credit: ESA/Hubble & NASA HST

Hot systems



Short pairwise interactions



→ **Gravitational wake**

Globular cluster (NGC 1781)

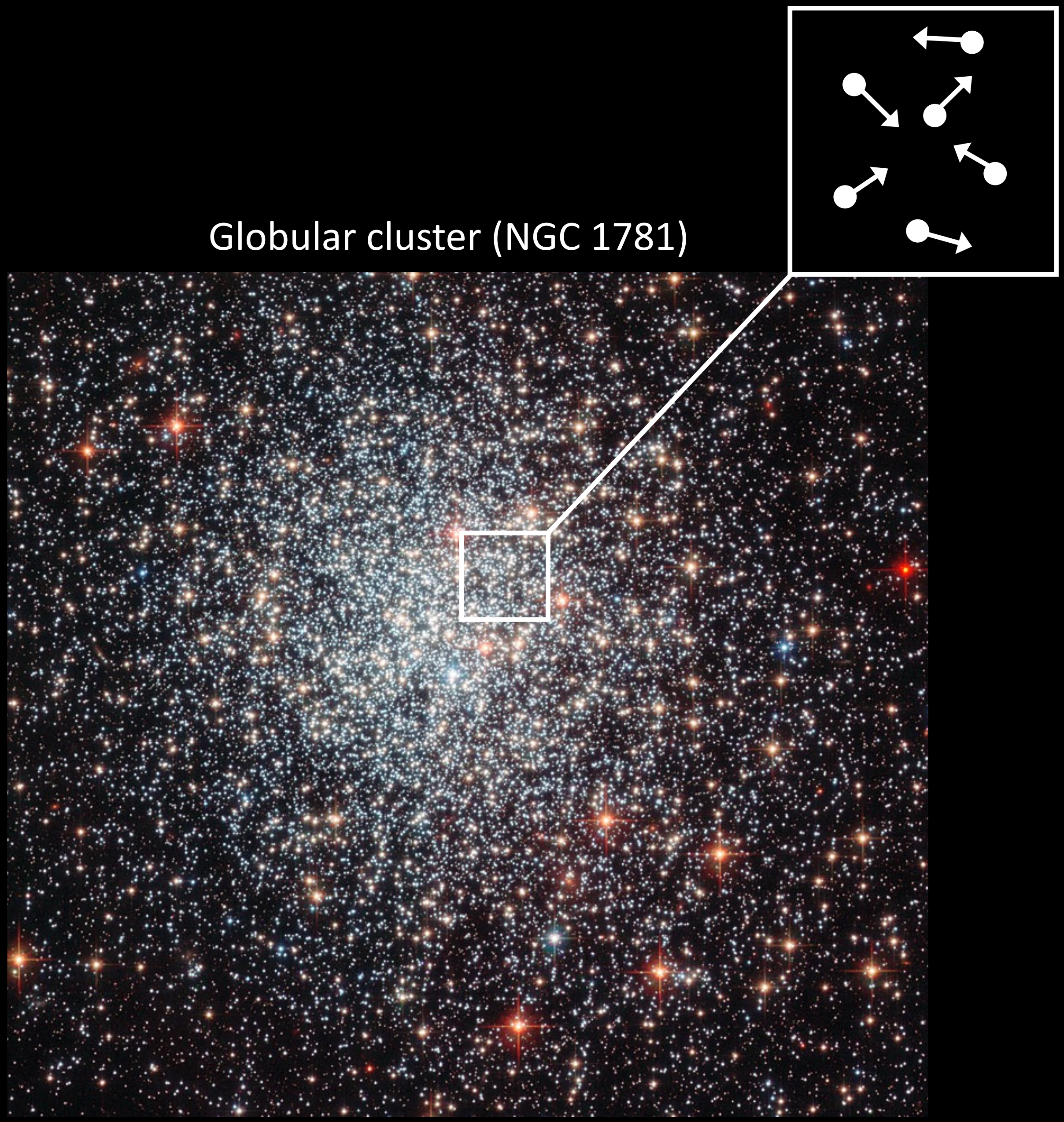
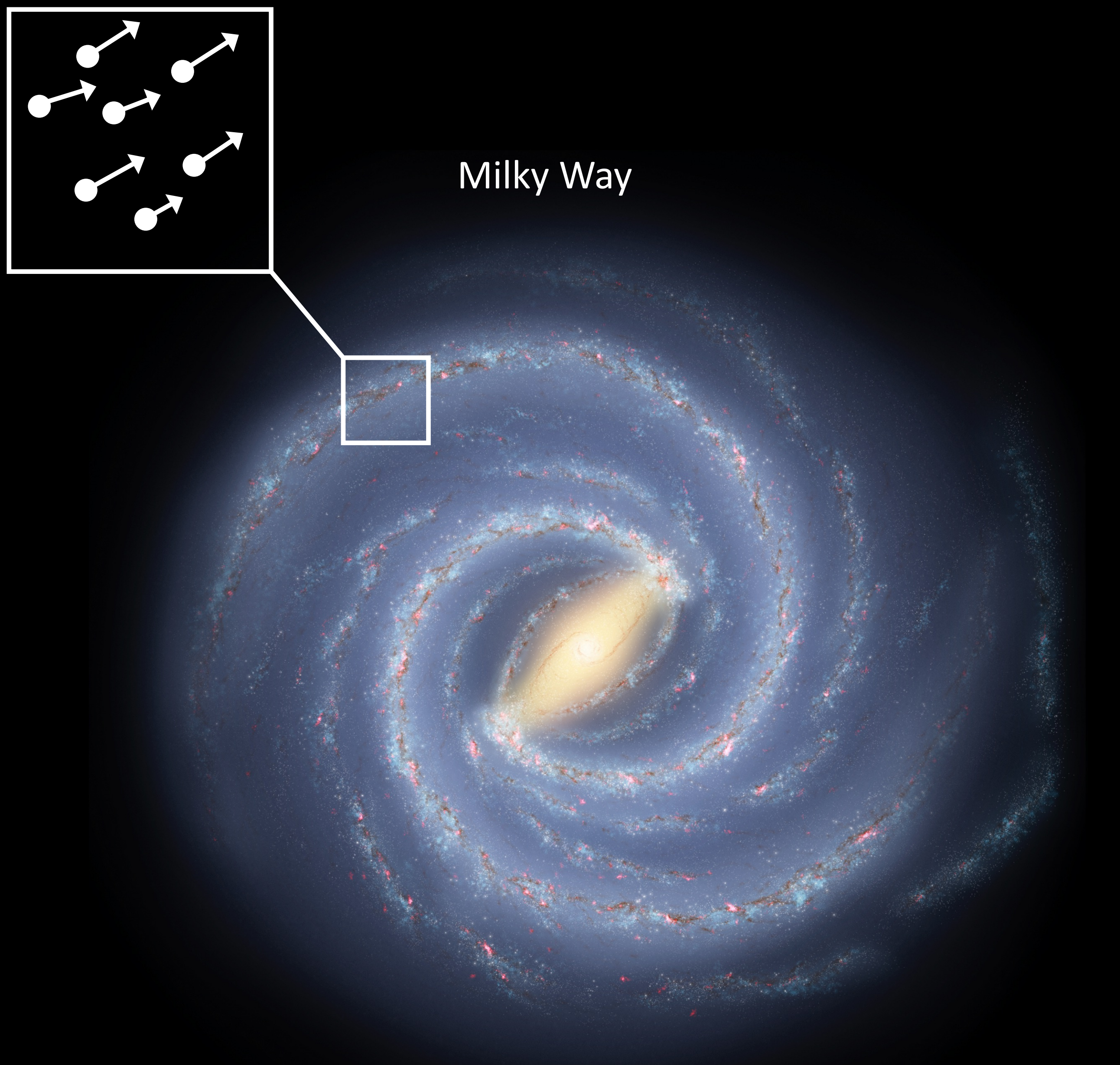


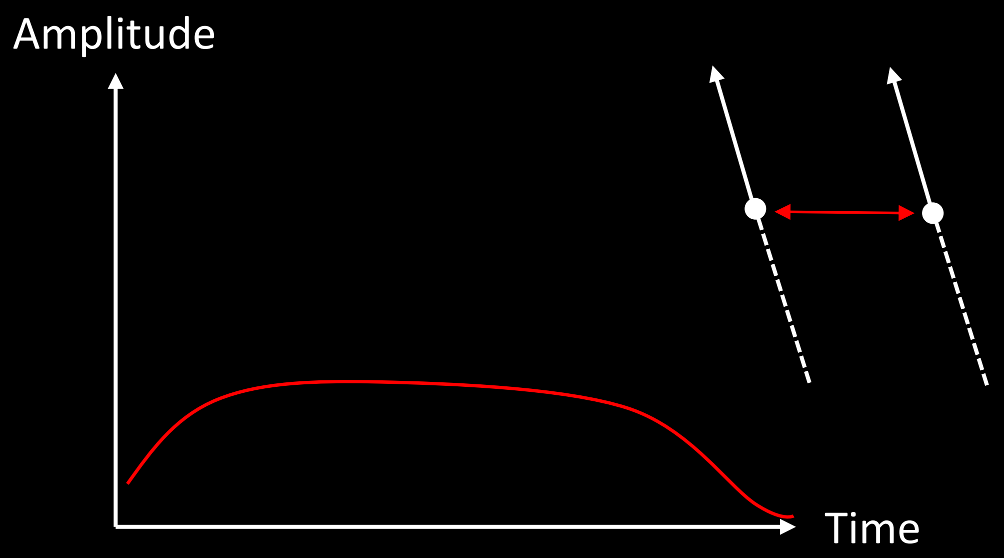
Image credit: ESA/Hubble & NASA HST

Cold systems

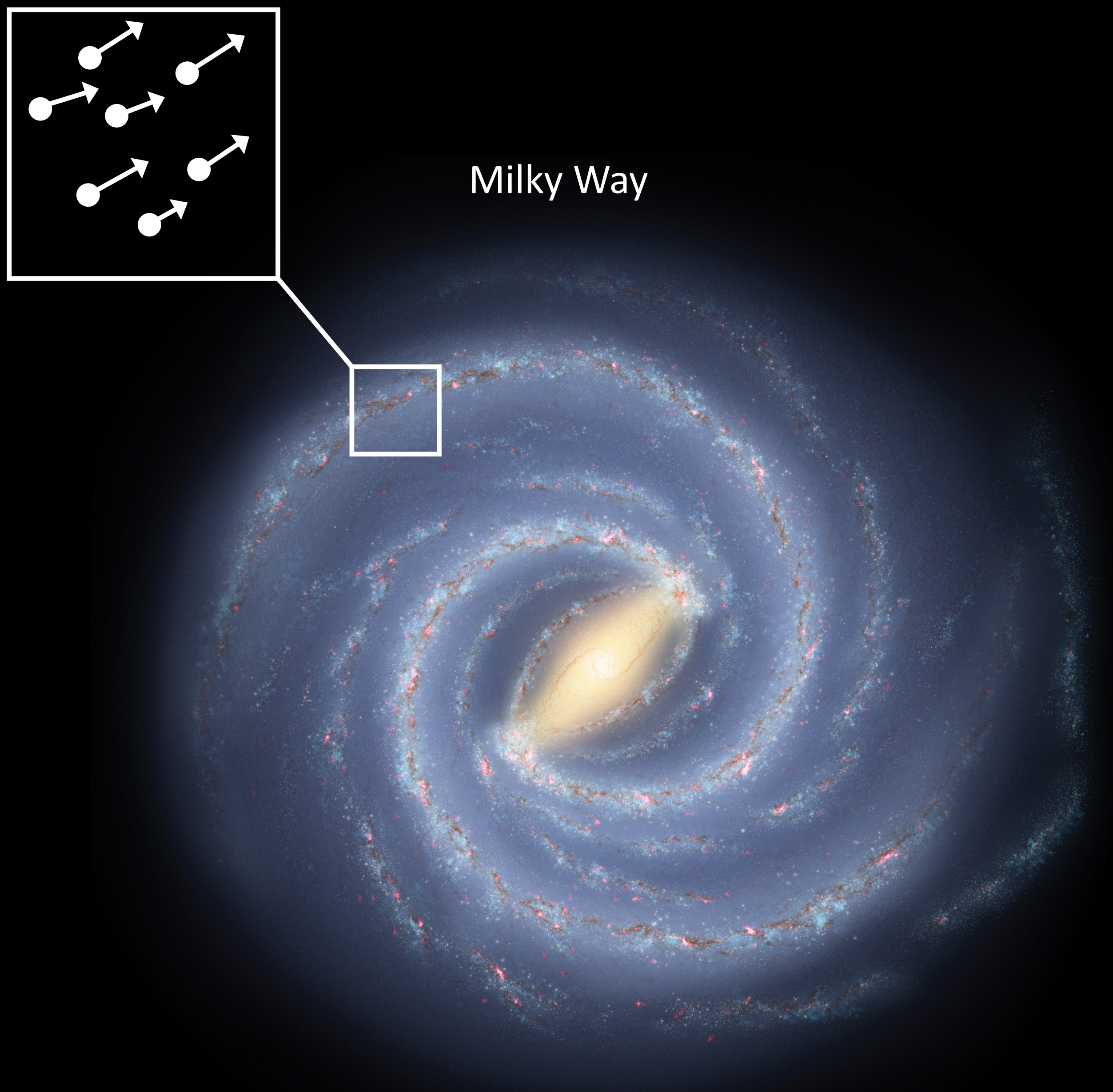


→ Supported by centrifugal force

Cold systems



Long coherent interaction build-up



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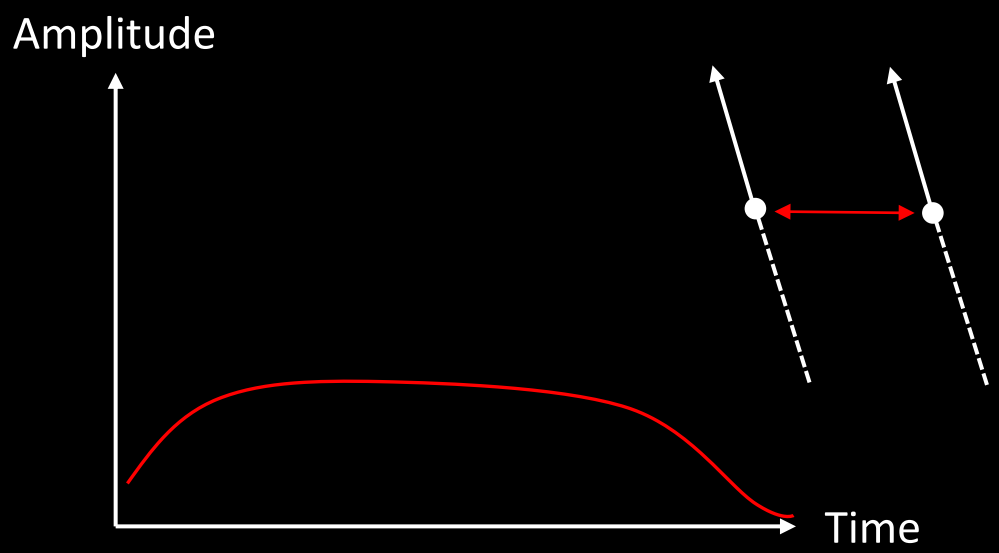

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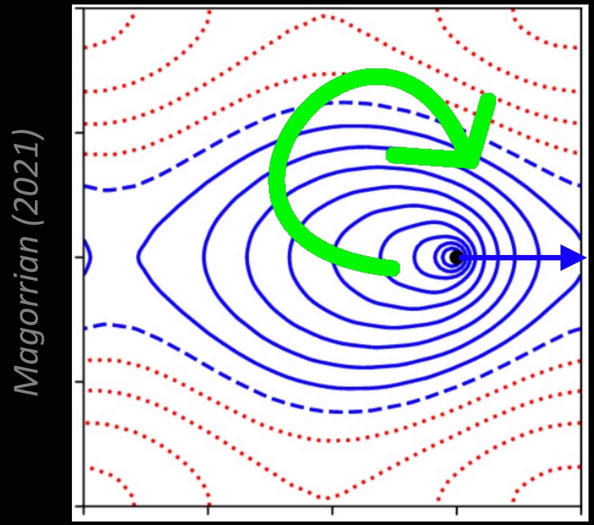
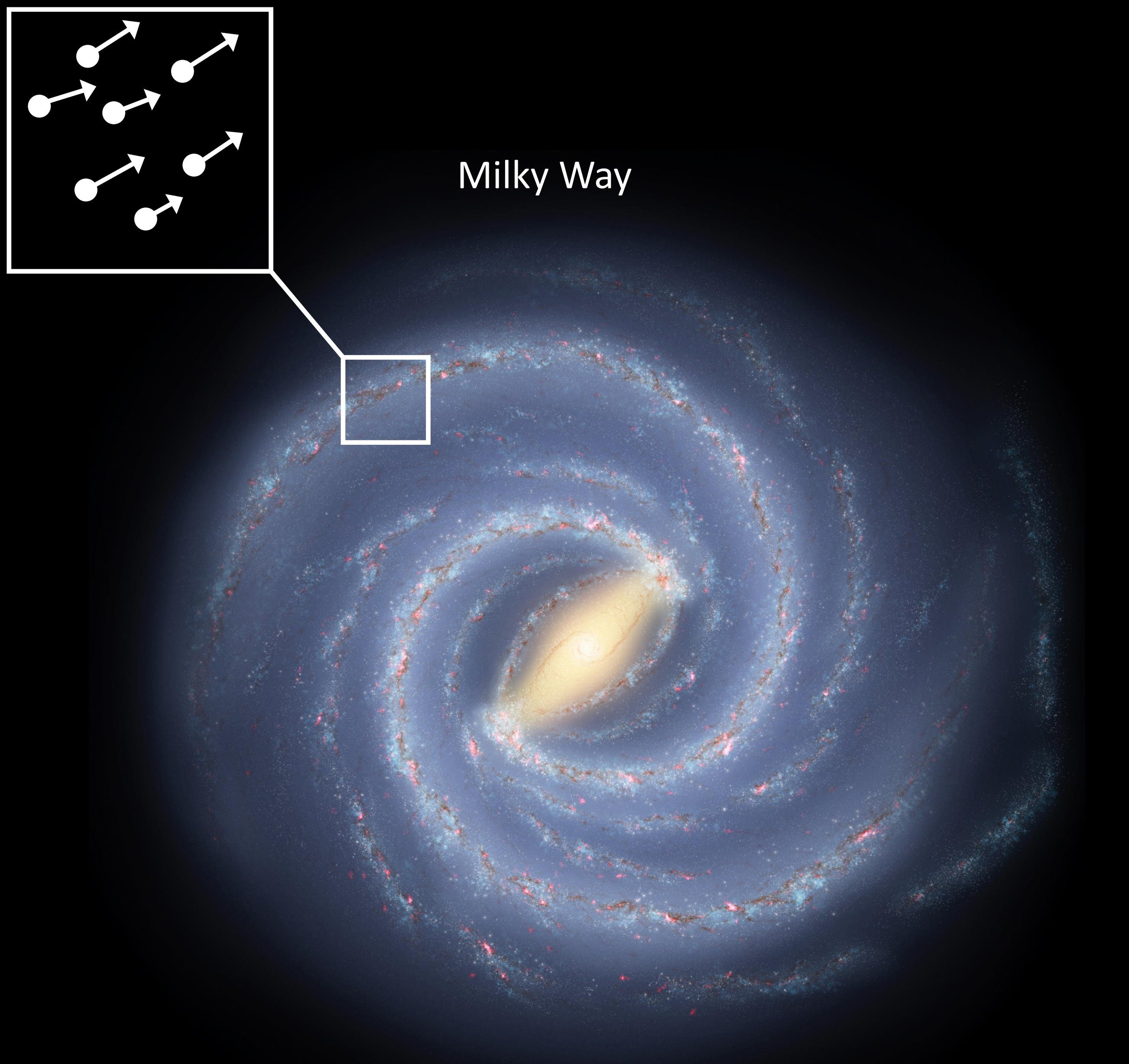


→ **<u>Self-amplified</u> gravitational wake**



→ Supported by centrifugal force

Predicting the secular fate of globular clusters

Credit: *NASA/ESA*

Messier 15 (HST)

- How to make theoretical predictions ?
- What mechanisms impact secular evolution?
- How does kinematics impact evolution ?
  (hot or cold)

Theoretical prediction

- Goal: evolution of the statistical ensemble of these objects

$$m_i \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = \sum_{j \neq i} \mathbf{F}_{j \to i}$$

- Costly, non-linear evolution

Credit: *NASA/ESA*

Messier 15 (HST)

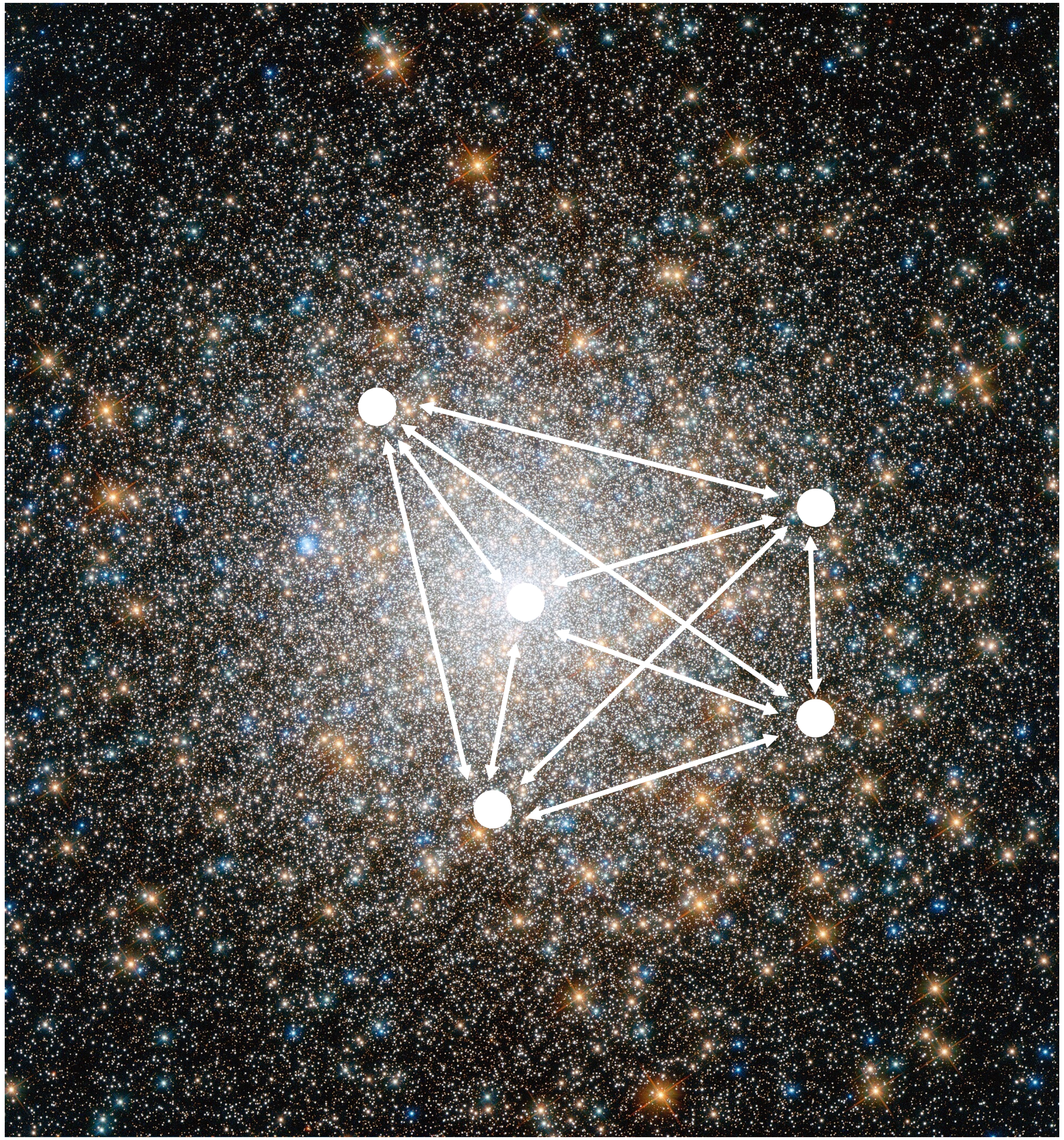
Theoretical prediction

- Goal: evolution of the statistical ensemble of these objects

$$m_i \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = \sum_{j \neq i} \mathbf{F}_{j \to i}$$

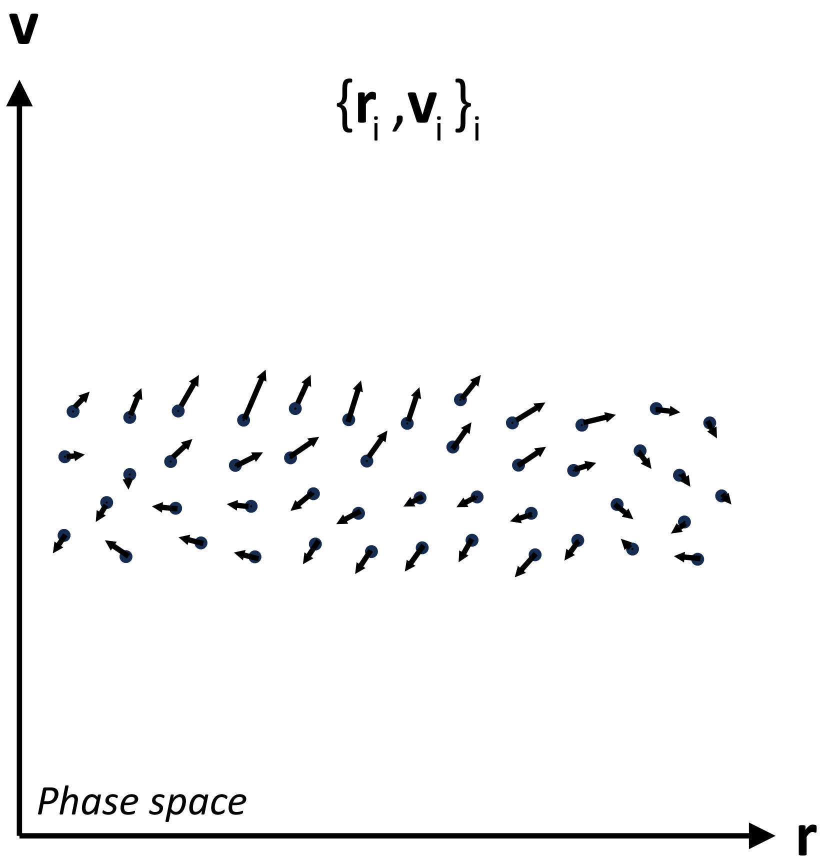
- Costly, non-linear evolution
- Gravity is long-range

Credit: *NASA/ESA*



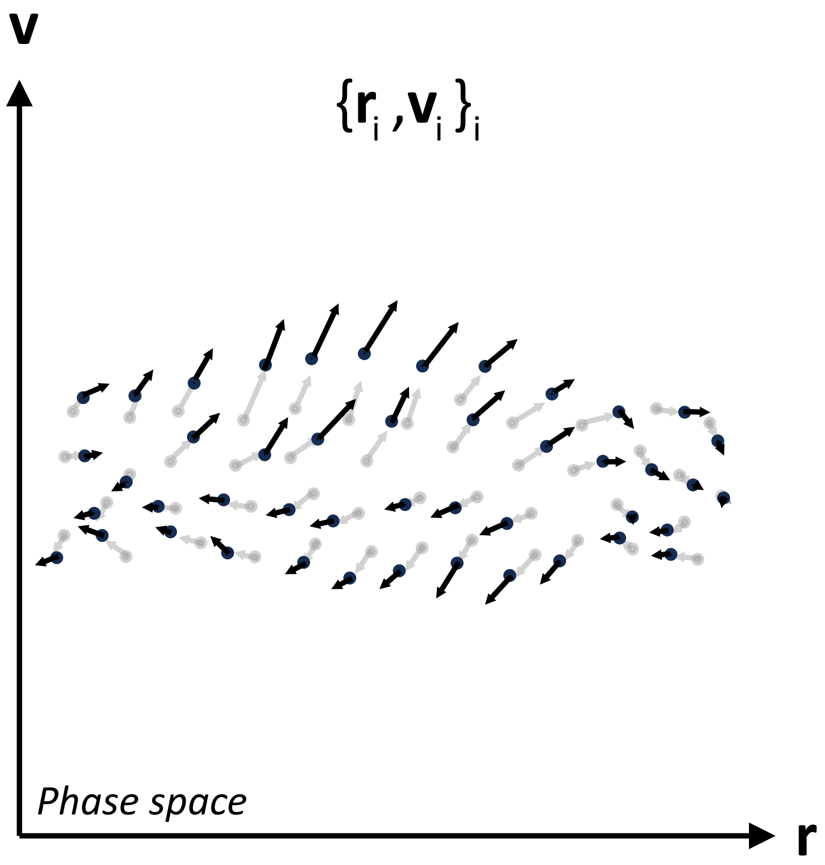
Messier 15 (HST)

Hamiltonian dynamics



Sampling of N position-velocities

Hamiltonian dynamics



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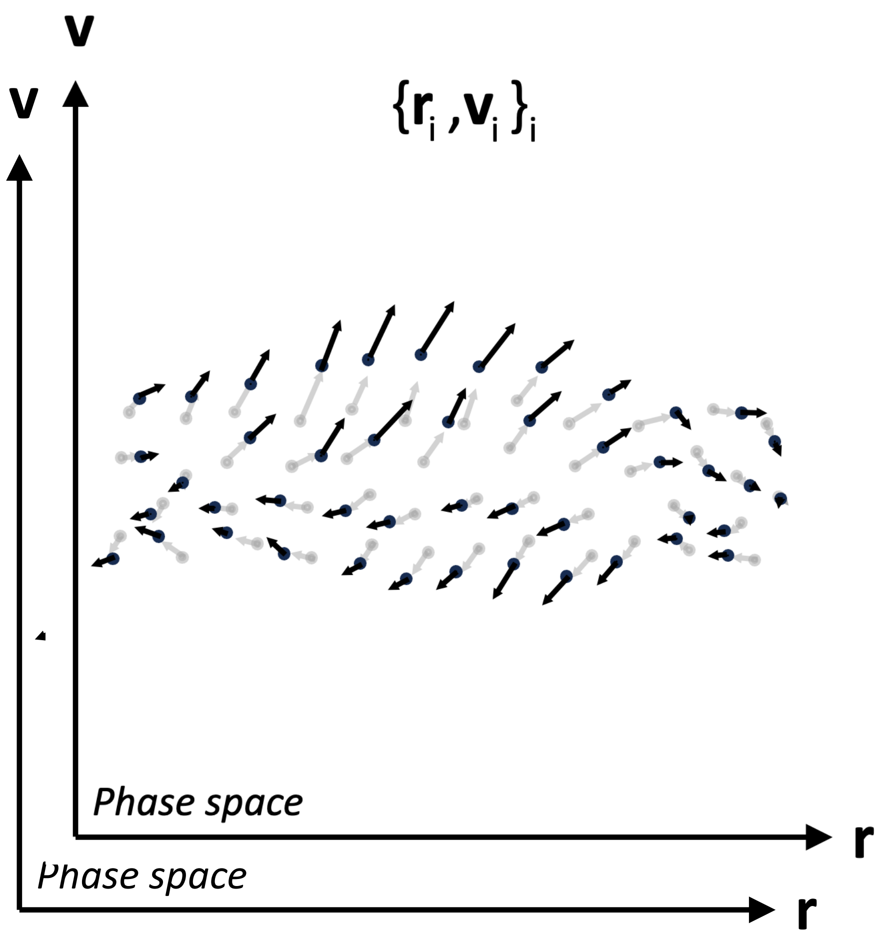

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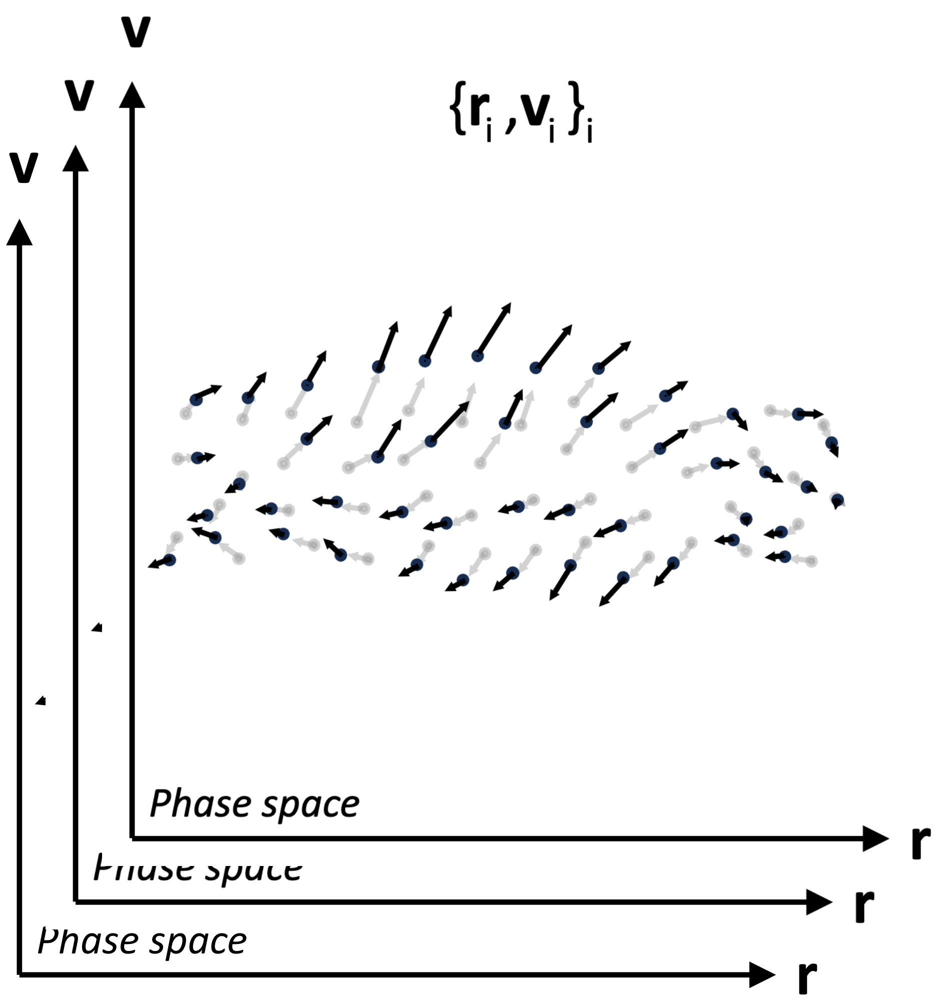

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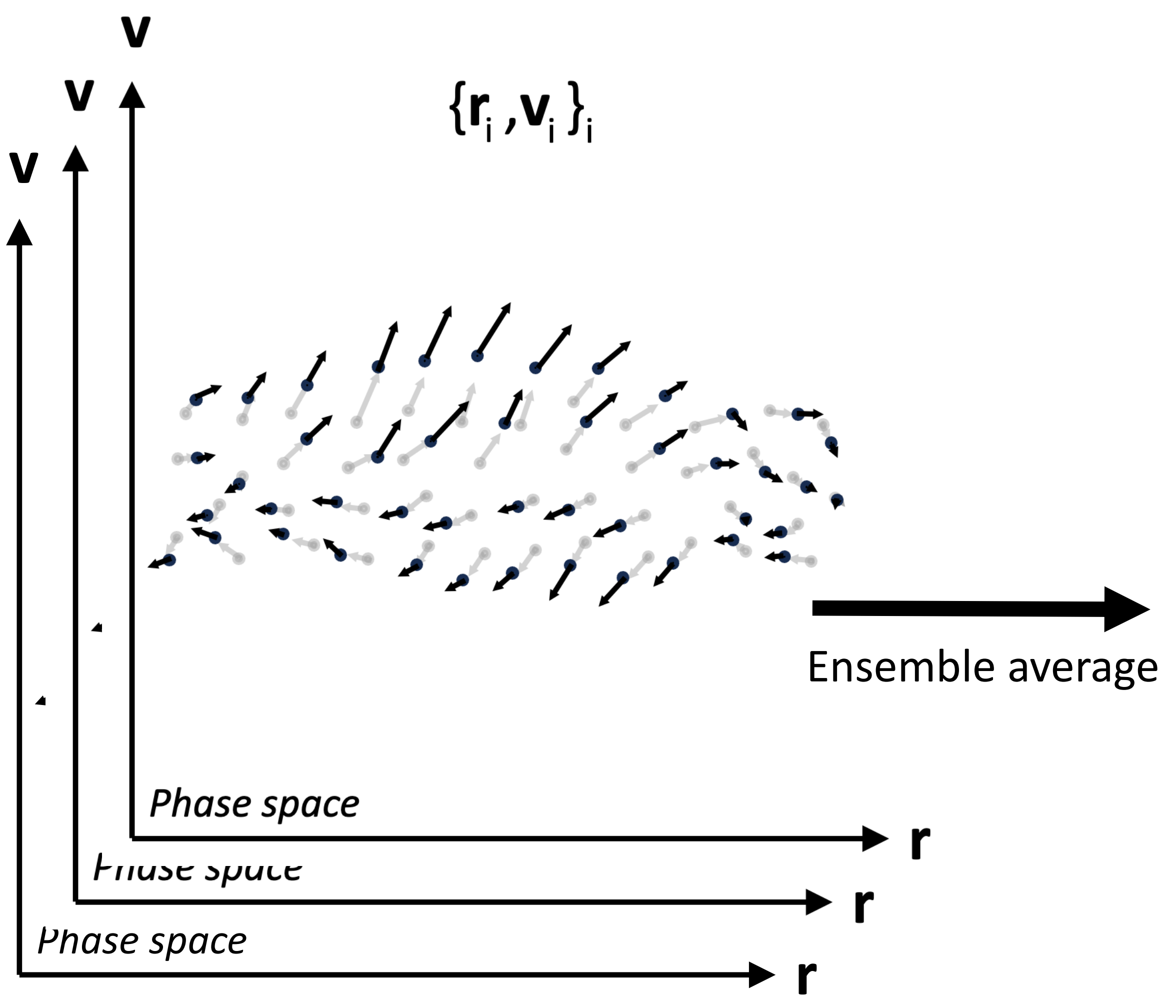

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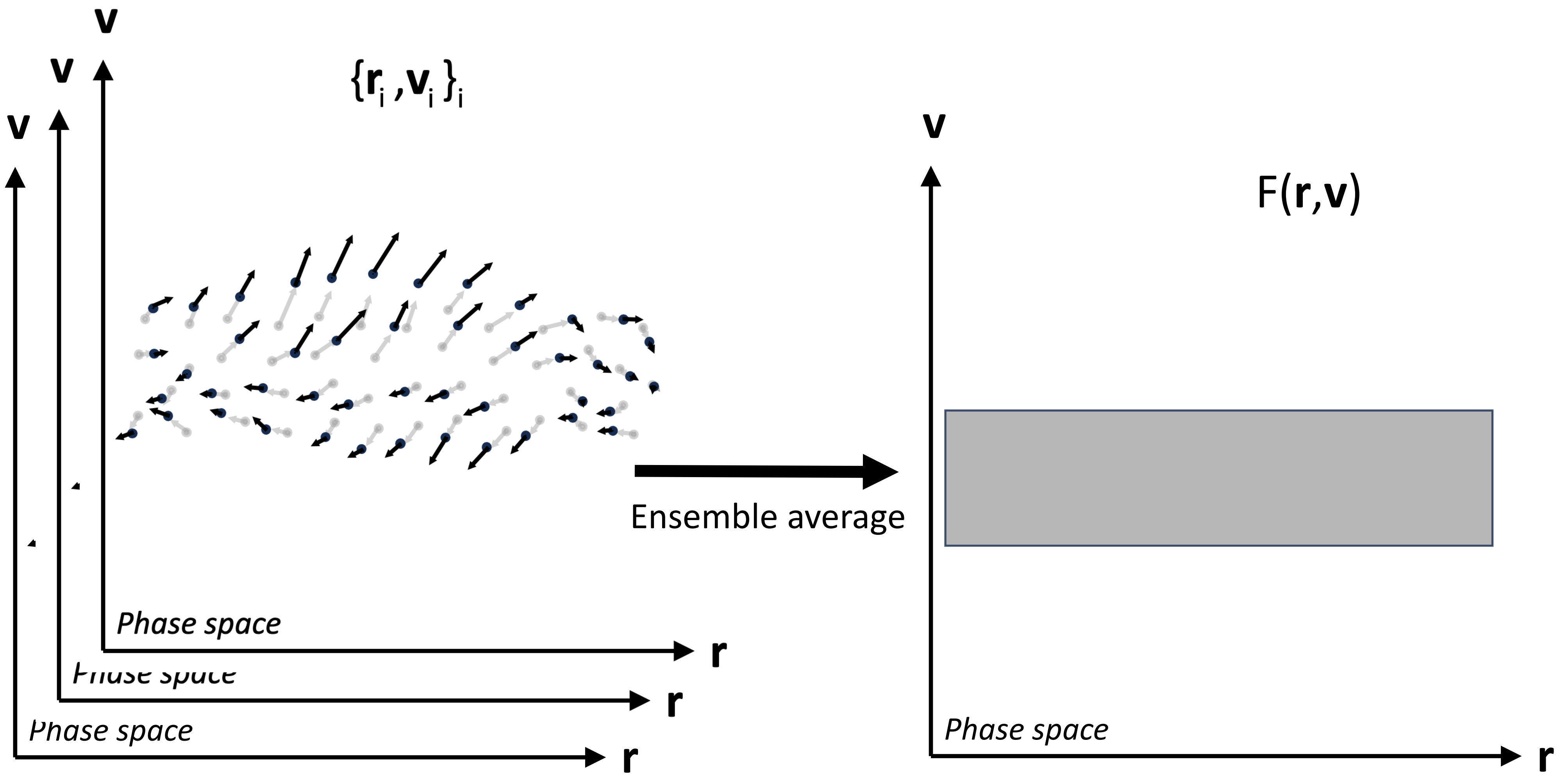
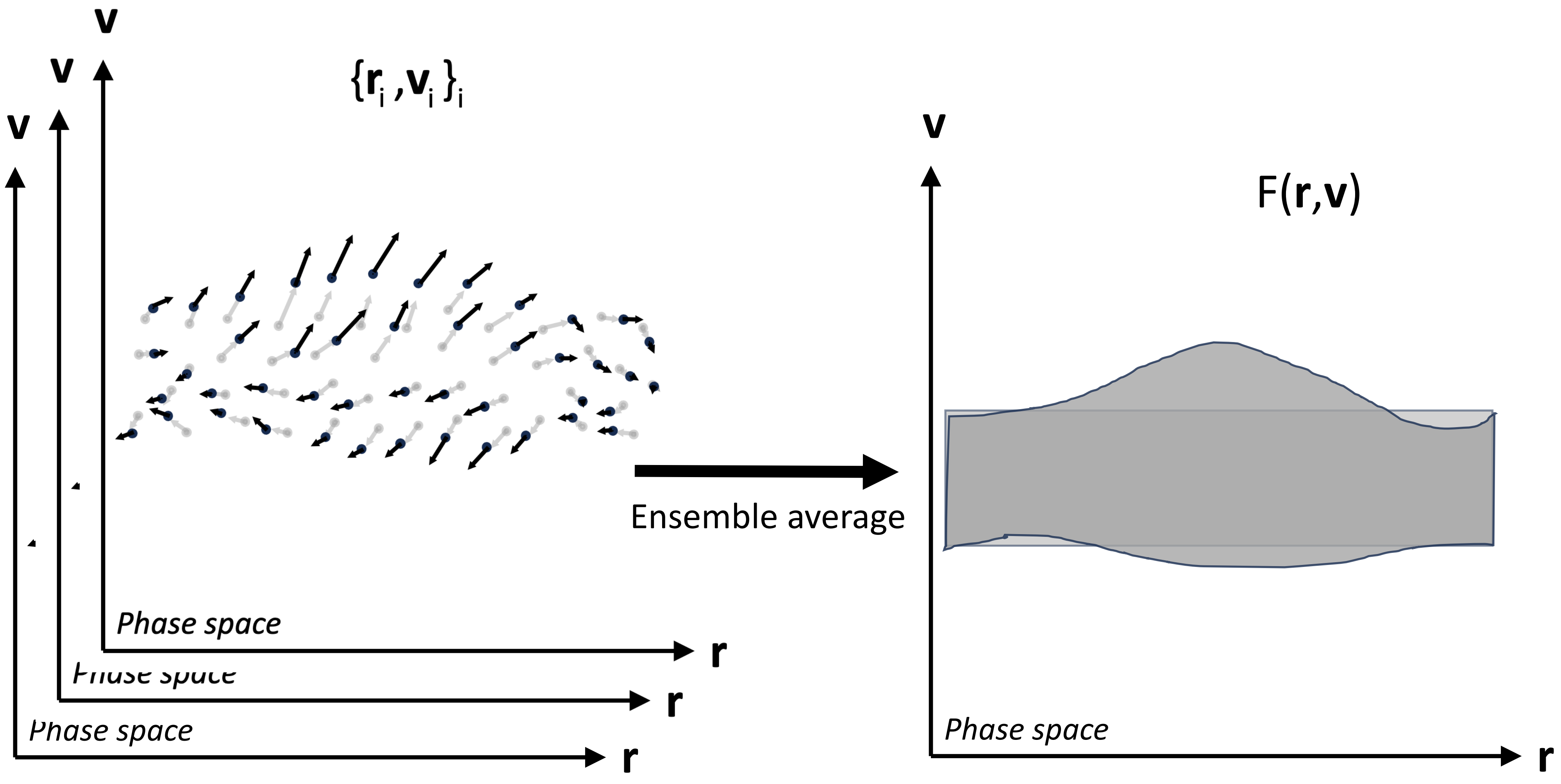


Sampling of N position-velocities

Continuous distribution function (DF)

Hamiltonian dynamics



Sampling of N position-velocities

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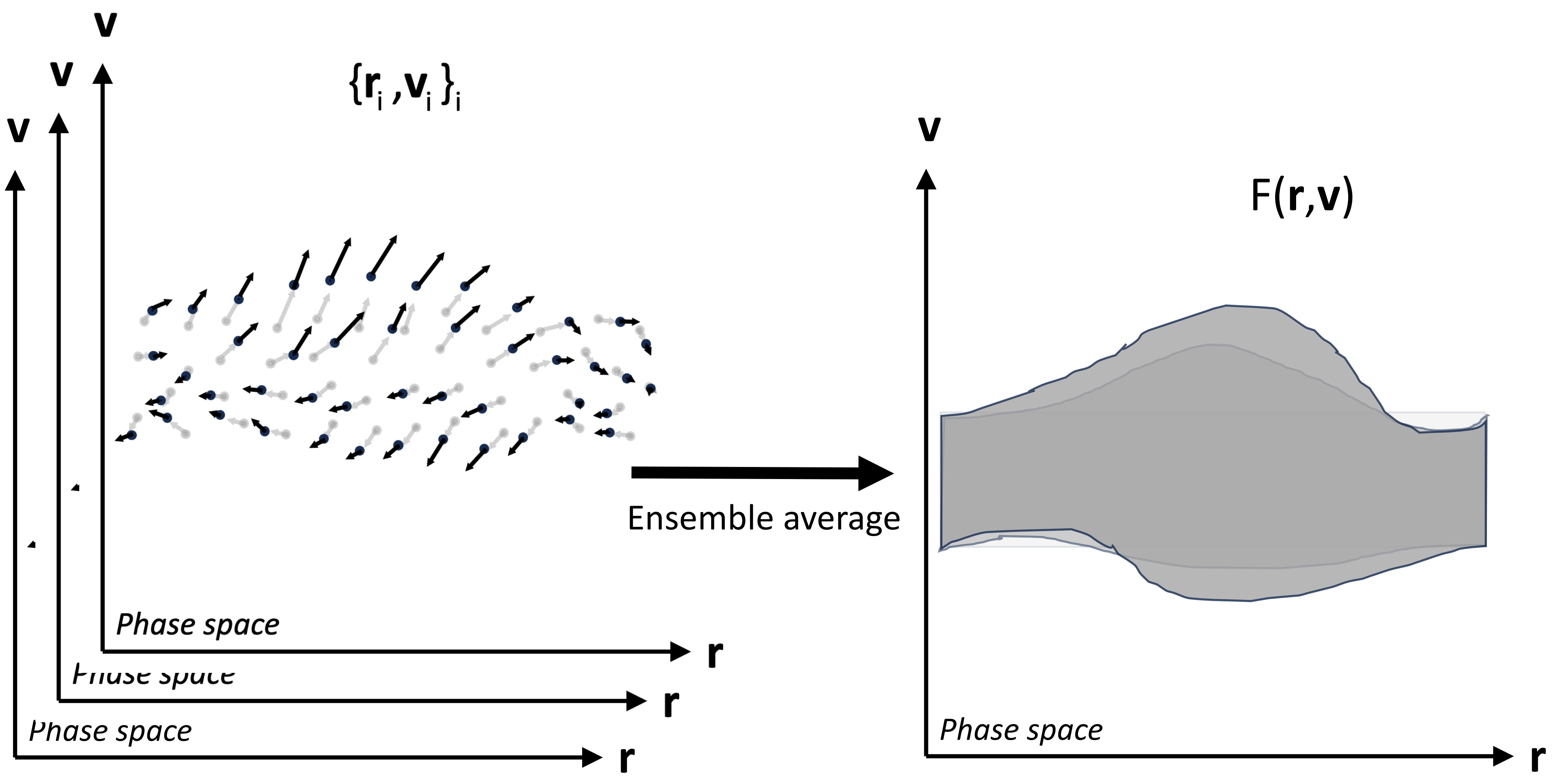

Sampling of N position-velocities

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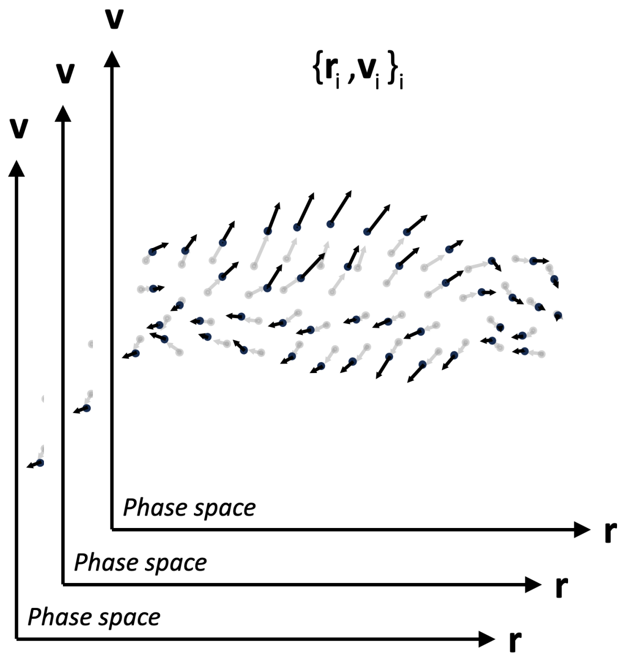
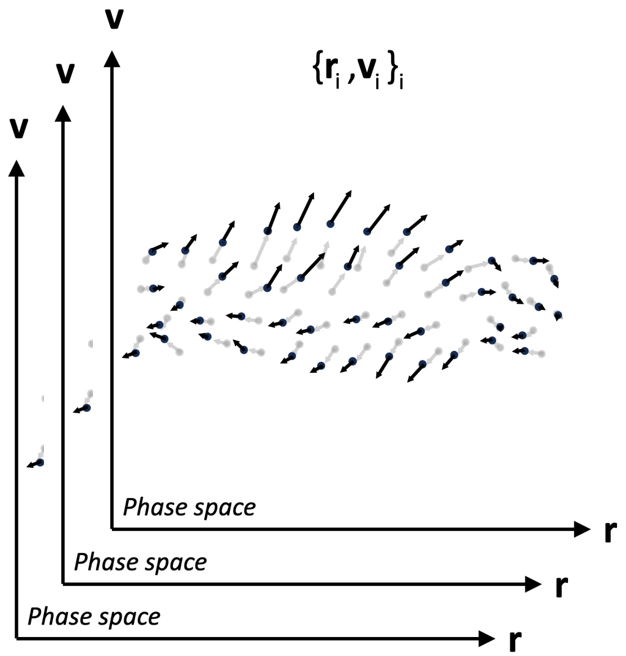


$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{r}_i}$$

$$H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 - \sum_{i<j} \frac{Gm}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

Hamiltonian dynamics



Newton's equations

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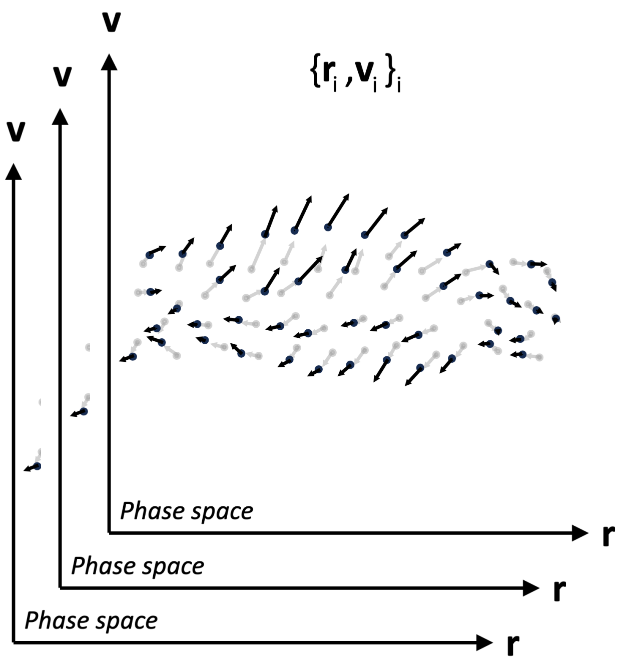
Hamiltonian dynamics



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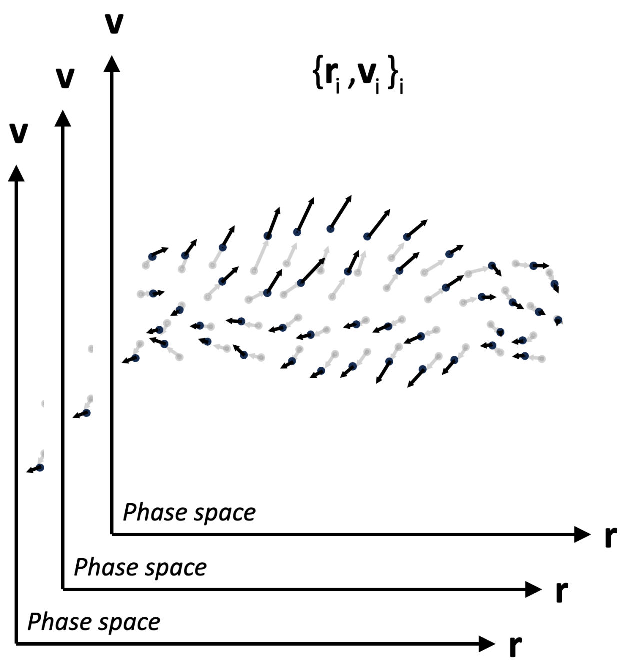
Hamiltonian: global invariant

Hamiltonian dynamics



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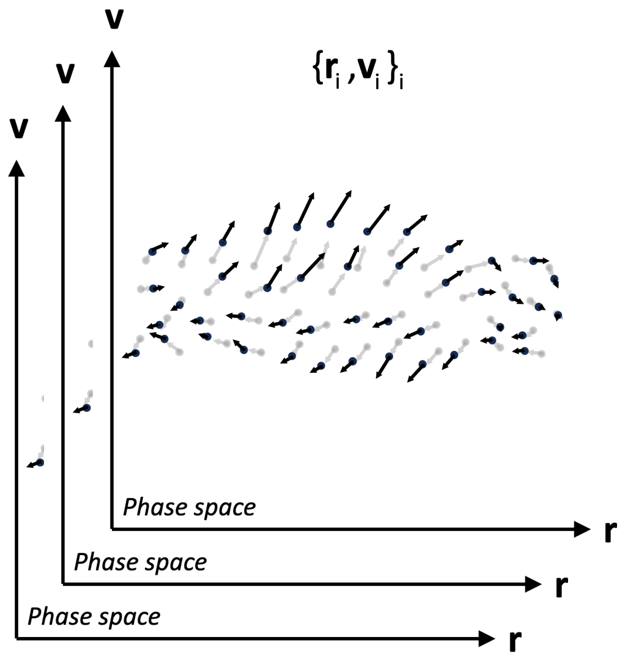
→ 6N equations times number of realisations

Hamiltonian dynamics

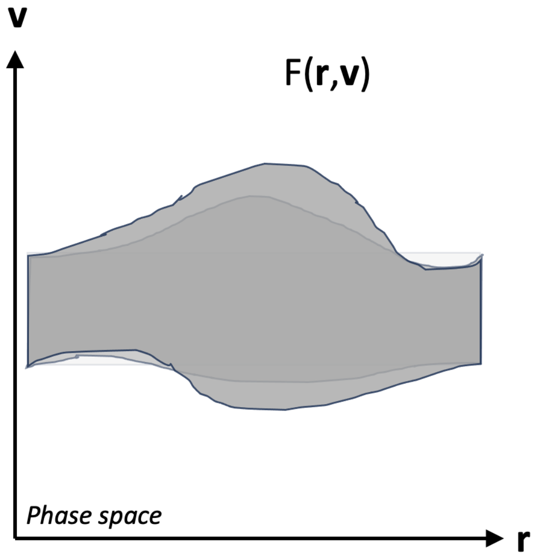


$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{r}_i}$$

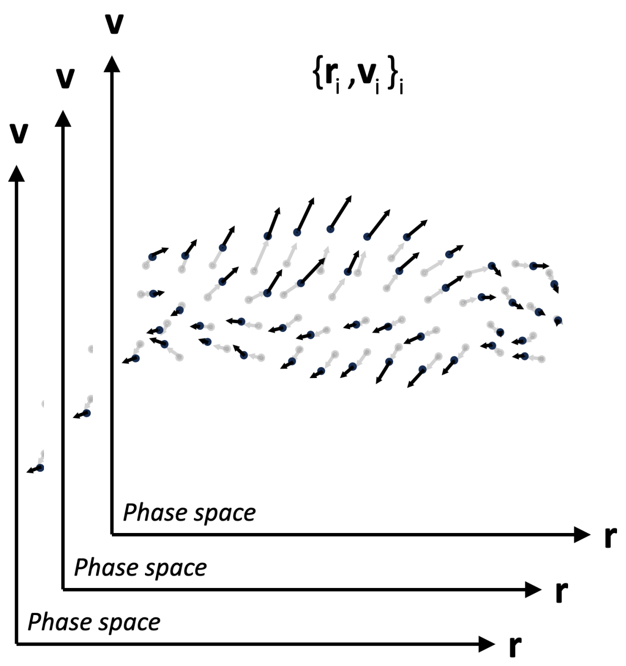
$$H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 - \sum_{i<j} \frac{Gm}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

→ 6N equations times number of realisations

Ensemble-averaged DF



Hamiltonian dynamics



$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{r}_i}$$

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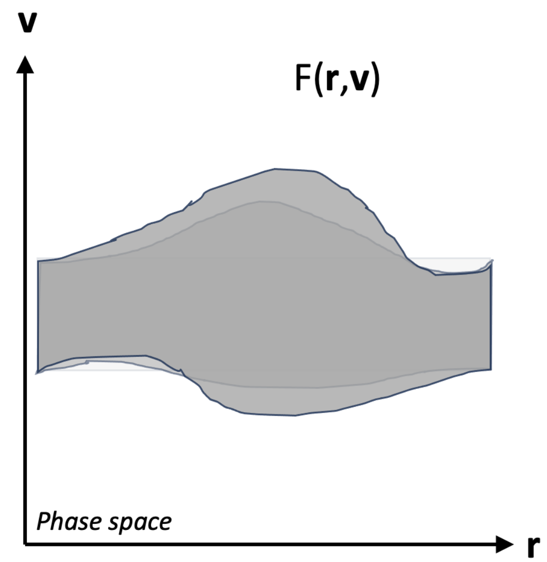
→ 6N equations times number of realisations

Ensemble-averaged DF

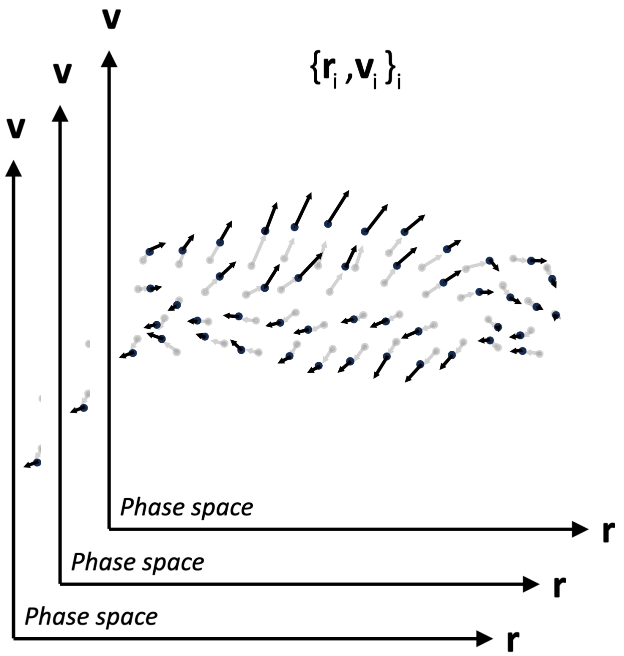


Continuity equation

$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial \boldsymbol{v}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{\partial H}{\partial \boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = \mathcal{C}[F]$$

$$H = \frac{1}{2}\boldsymbol{v}^2 + \Phi[F](\boldsymbol{r})$$

Hamiltonian dynamics

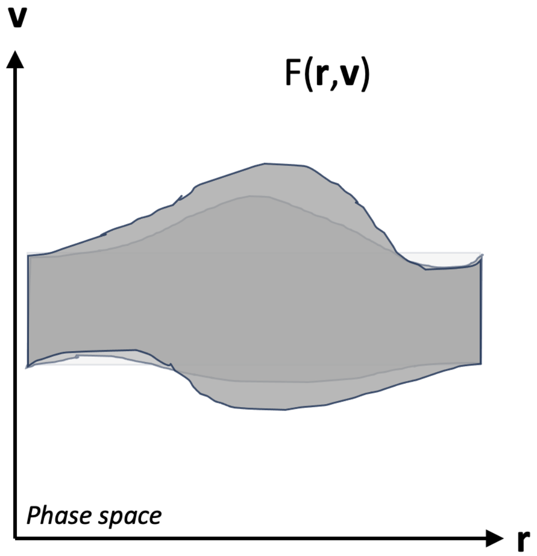


$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{r}_i}$$

$$H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 - \sum_{i<j} \frac{Gm}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

→ 6N equations times number of realisations

Ensemble-averaged DF

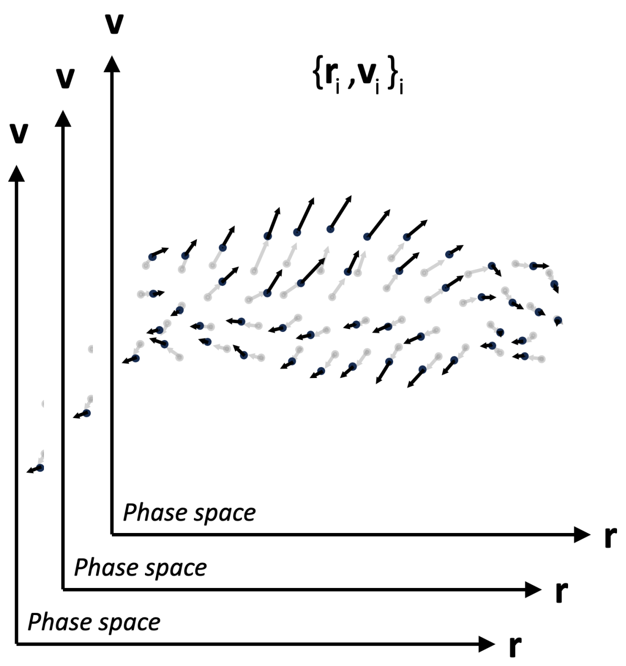


$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial \boldsymbol{v}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{\partial H}{\partial \boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = \mathcal{C}[F]$$

$$H = \frac{1}{2}\boldsymbol{v}^2 + \Phi[F](\boldsymbol{r})$$

Mean field Hamiltonian

Hamiltonian dynamics

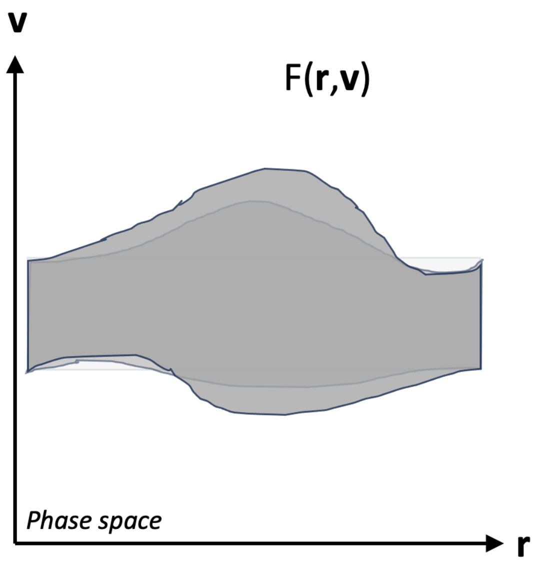


$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{r}_i}$$

$$H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 - \sum_{i<j} \frac{Gm}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

→ 6N equations times number of realisations

Ensemble-averaged DF

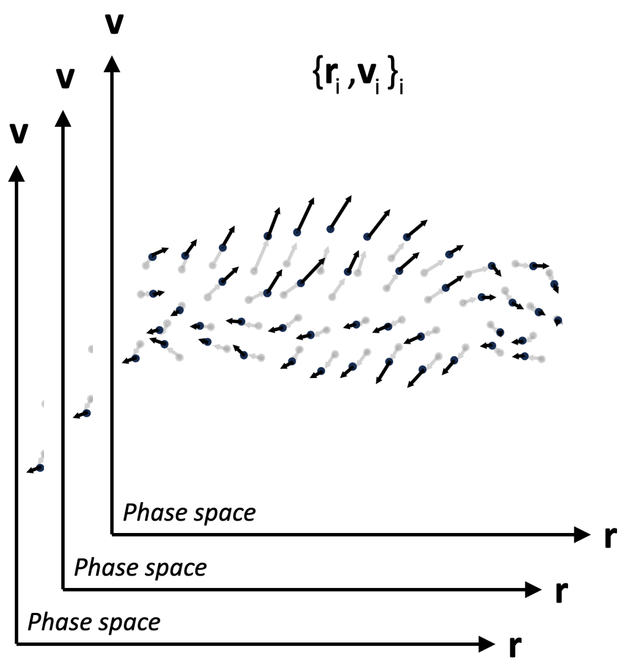


$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial \boldsymbol{v}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{\partial H}{\partial \boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = \mathcal{C}[F]$$

$$H = \frac{1}{2}\boldsymbol{v}^2 + \boxed{\Phi[F](\boldsymbol{r})}$$

Mean field

Hamiltonian dynamics

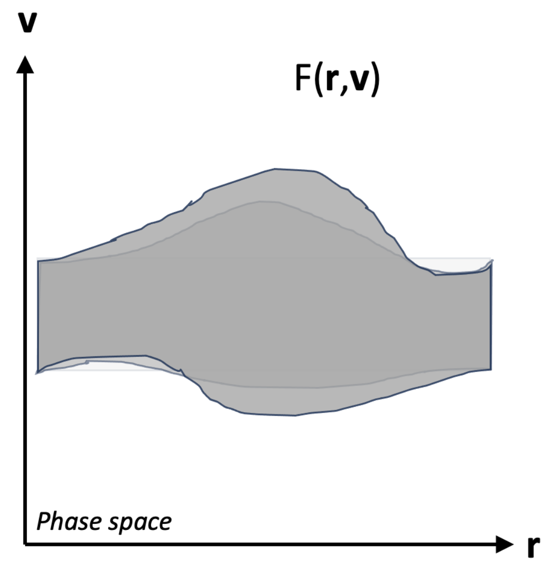


$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{r}_i}$$

$$H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 - \sum_{i<j} \frac{Gm}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

→ 6N equations times number of realisations

Ensemble-averaged DF

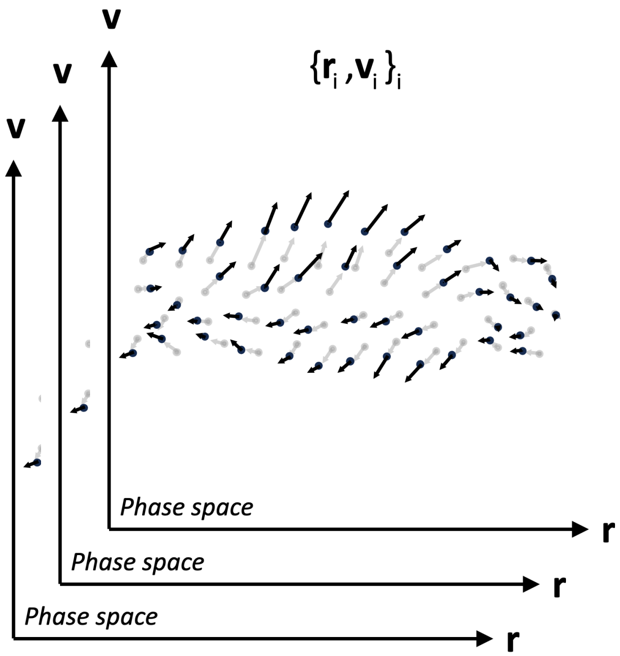


$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial \boldsymbol{v}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{\partial H}{\partial \boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = \mathcal{C}[F]$$

$$H = \frac{1}{2}\boldsymbol{v}^2 + \Phi[F](\boldsymbol{r})$$

→ 1 equation on the field

Mean field limit

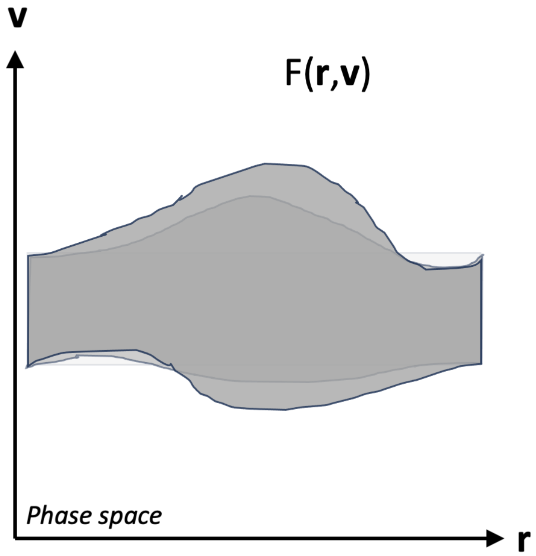


$$\frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{r}_i}$$

$$H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 - \sum_{i<j} \frac{Gm}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

→ 6N equations times number of realisations

Ensemble-averaged DF



$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial \boldsymbol{v}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{\partial H}{\partial \boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = \mathcal{C}[F]$$

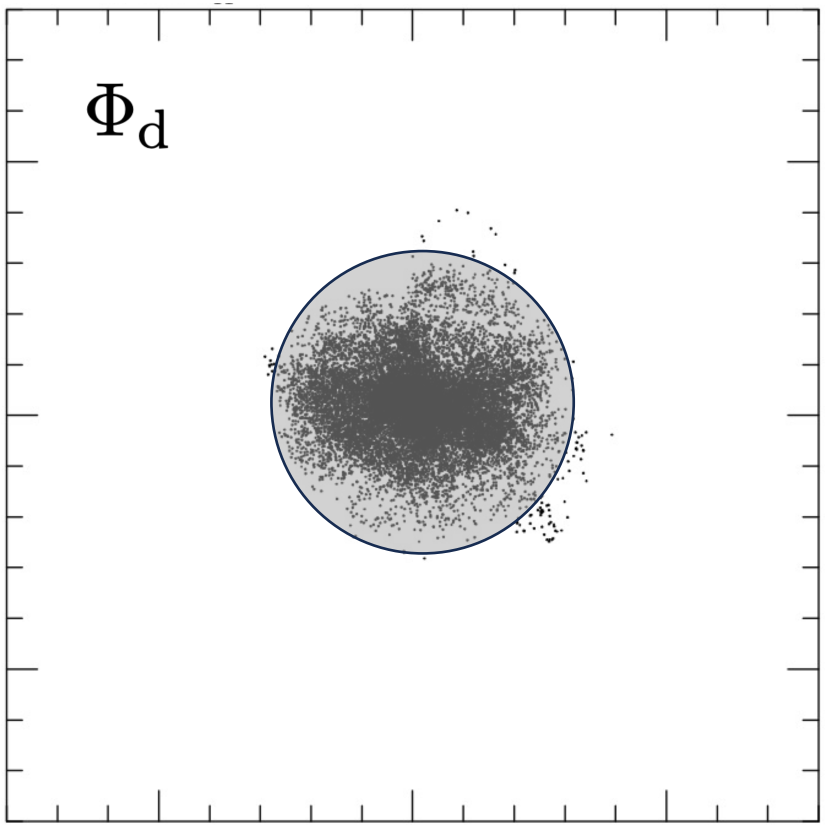
Mean-field limit → $\mathcal{C}[F] \to 0$

$$H = \frac{1}{2}\boldsymbol{v}^2 + \Phi[F](\boldsymbol{r})$$

→ 1 equation on the field

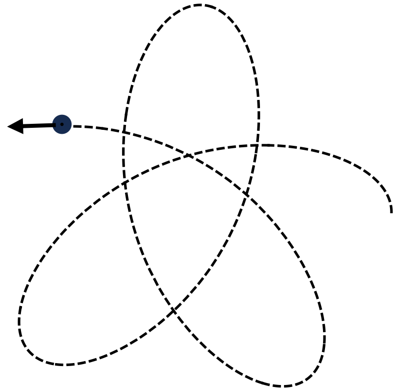
Angle action coordinates

$$\Phi_{\mathrm{d}} = \Phi + \delta\Phi$$



Binnet & Tremaine (2008)

Mean field

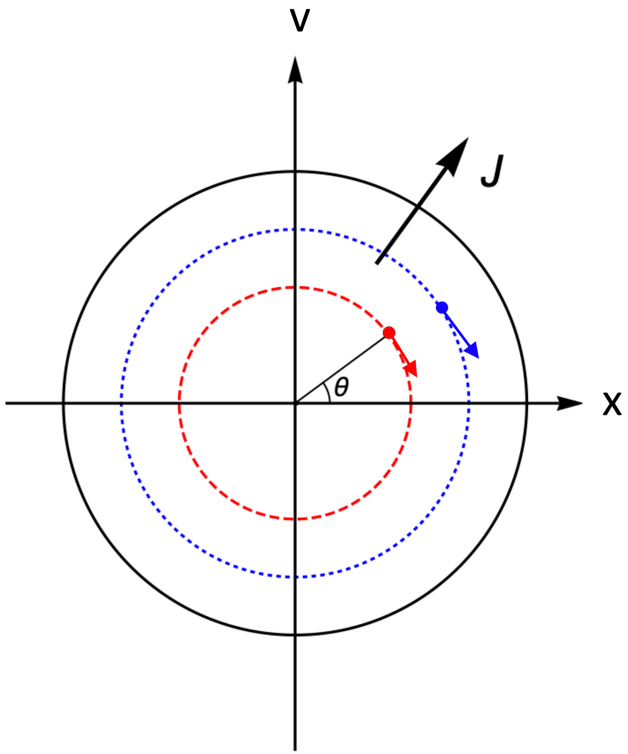


→ Symmetry of QSS

→ Orbit labelling: actions **J**

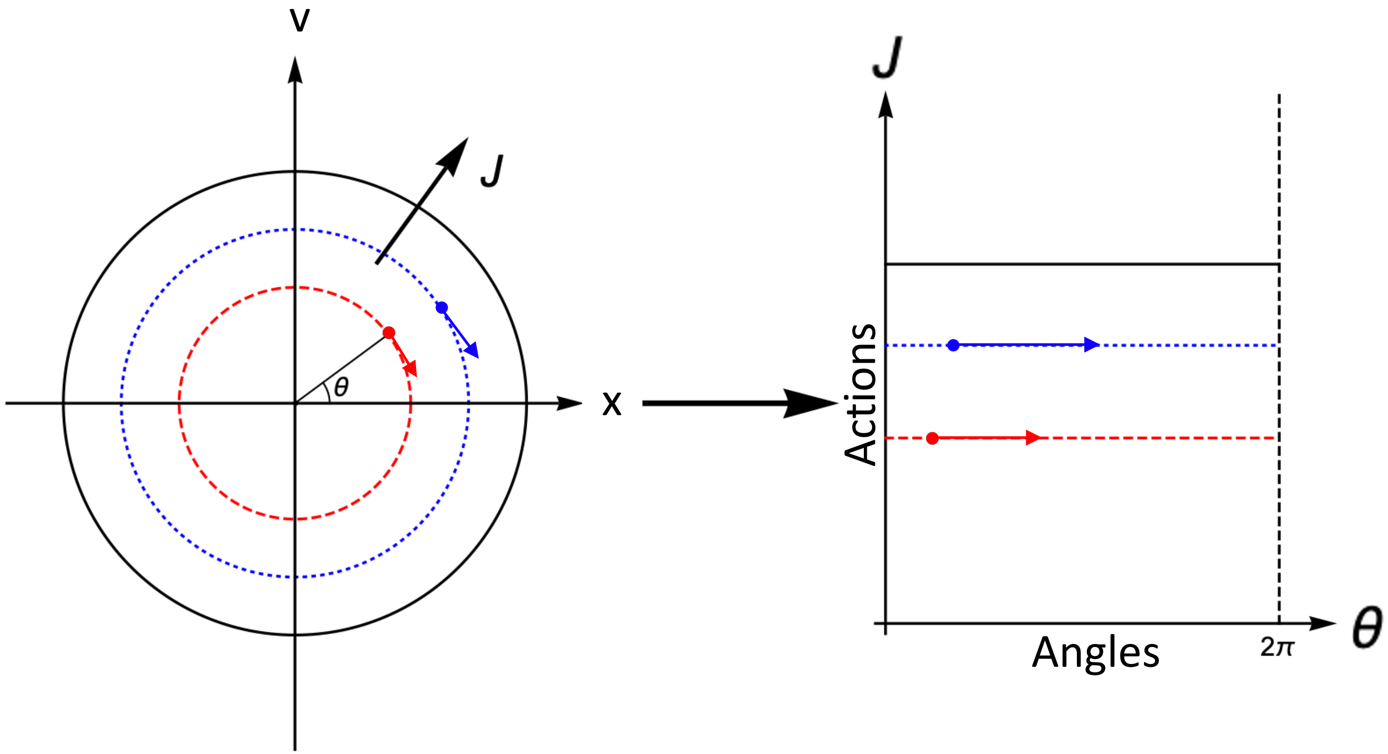
Angle action coordinates

- Action : motion integrals

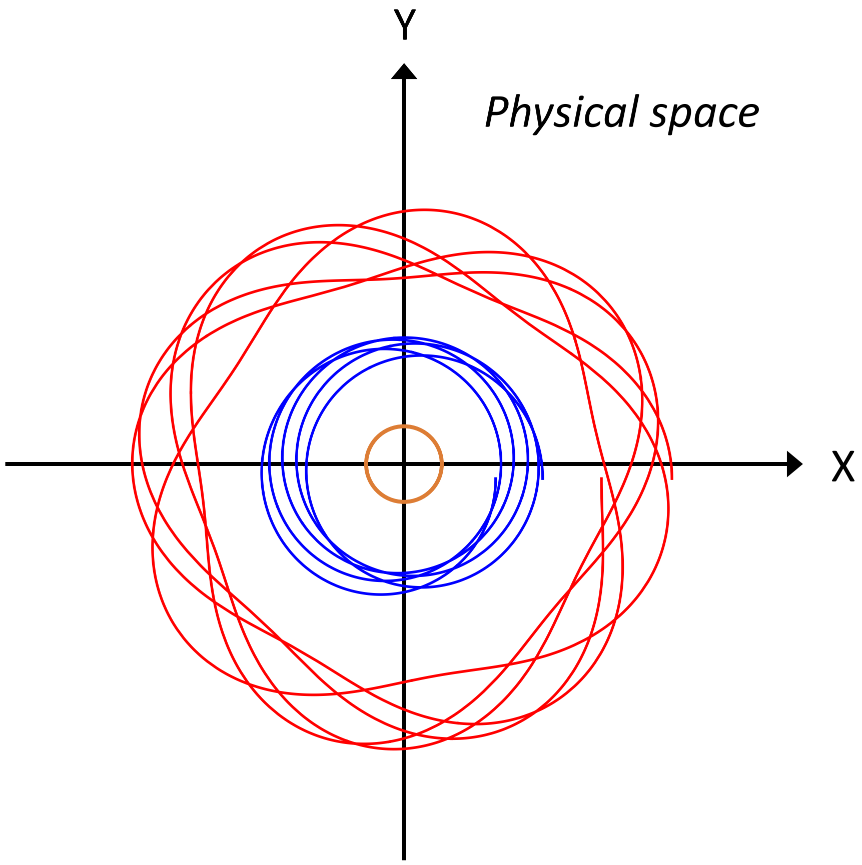


Angle action coordinates

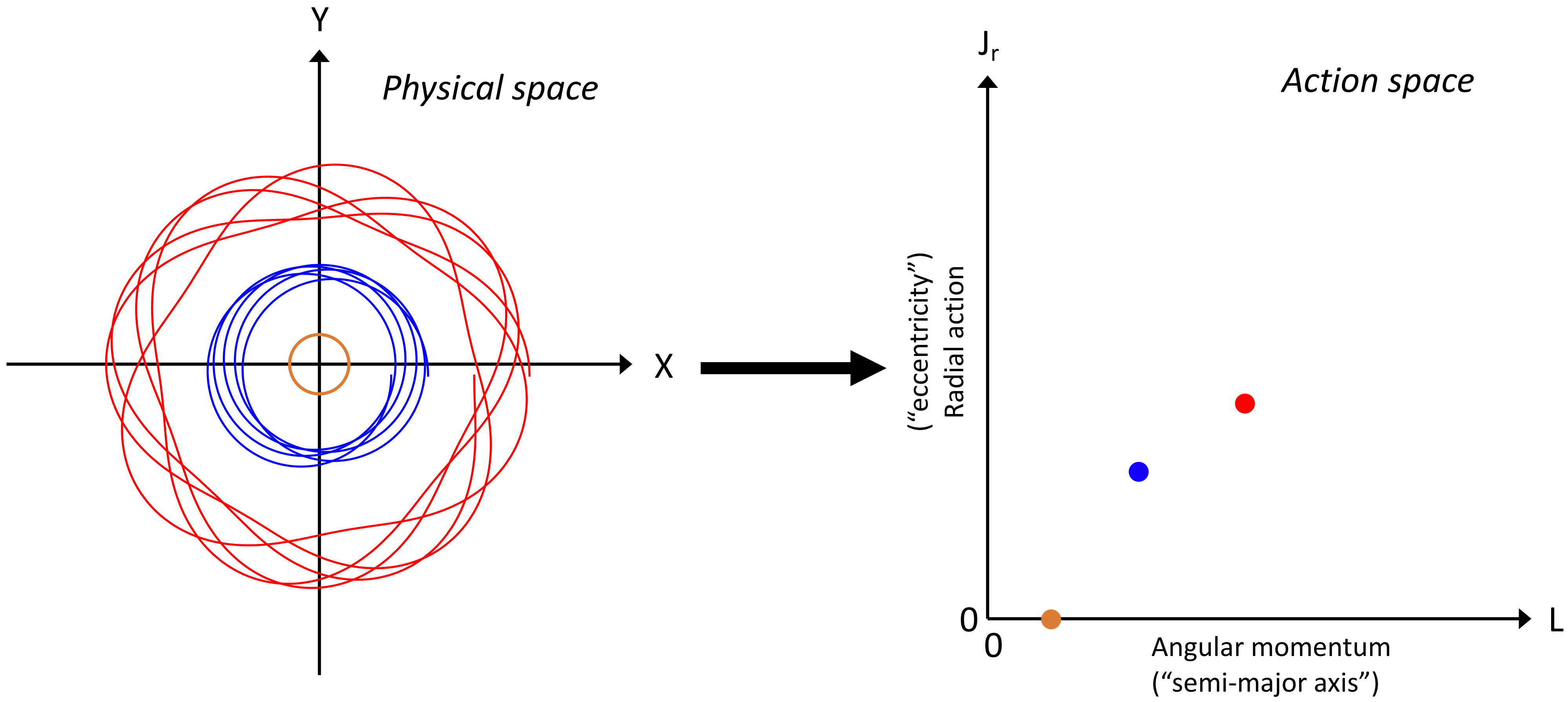
- Action : motion integrals



Actions in a globular cluster

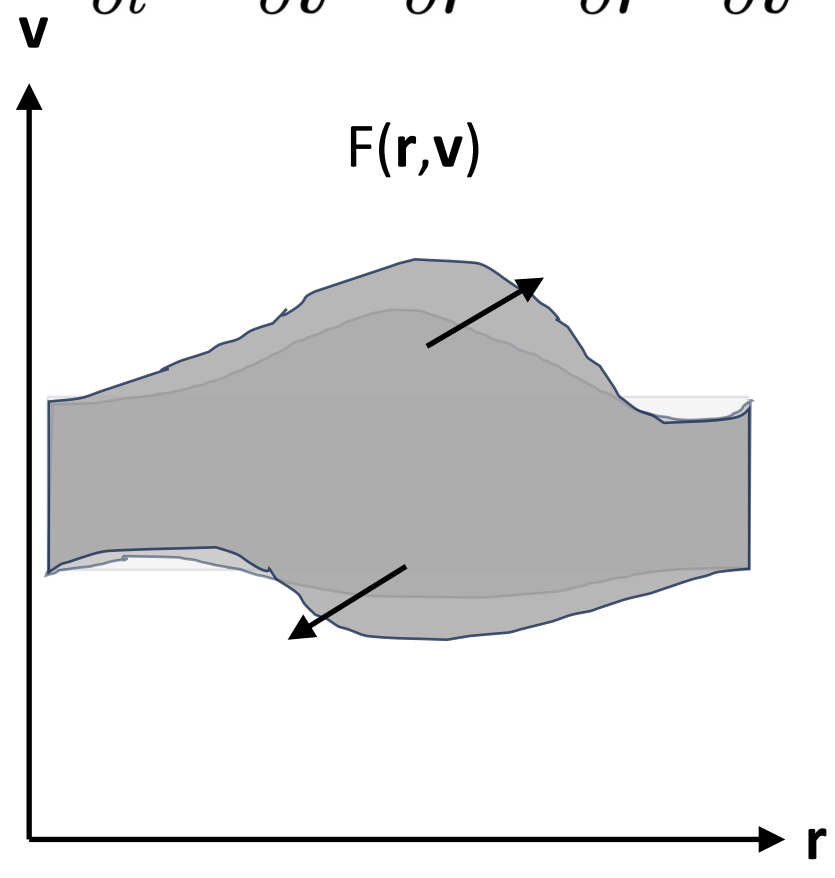


Actions in a globular cluster



Phase mixing

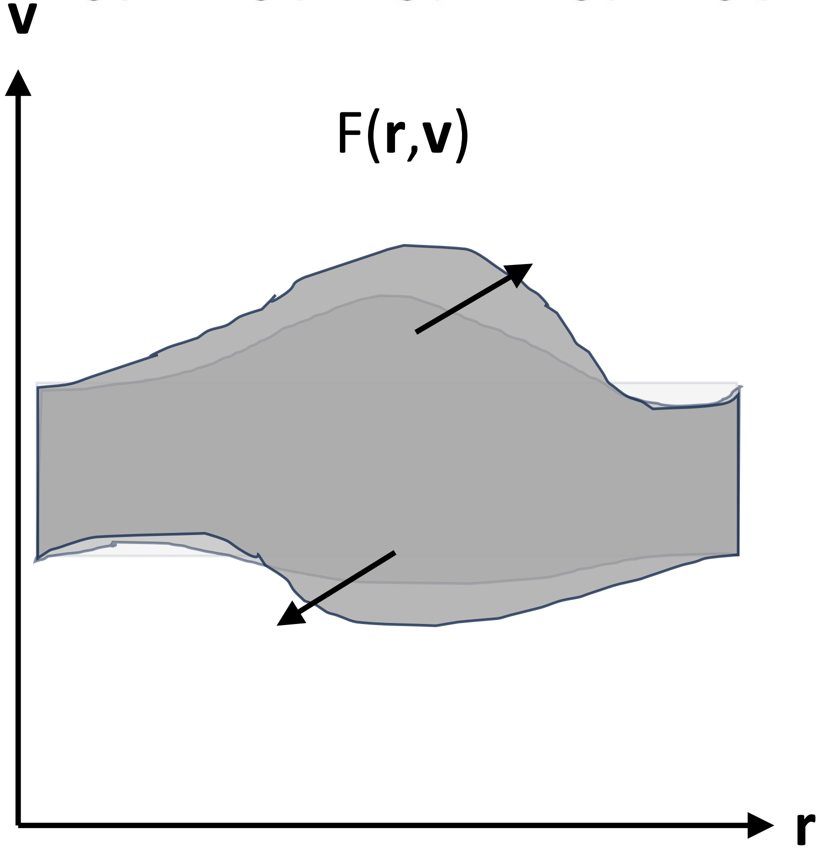
$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial \boldsymbol{v}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{\partial H}{\partial \boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = 0$$



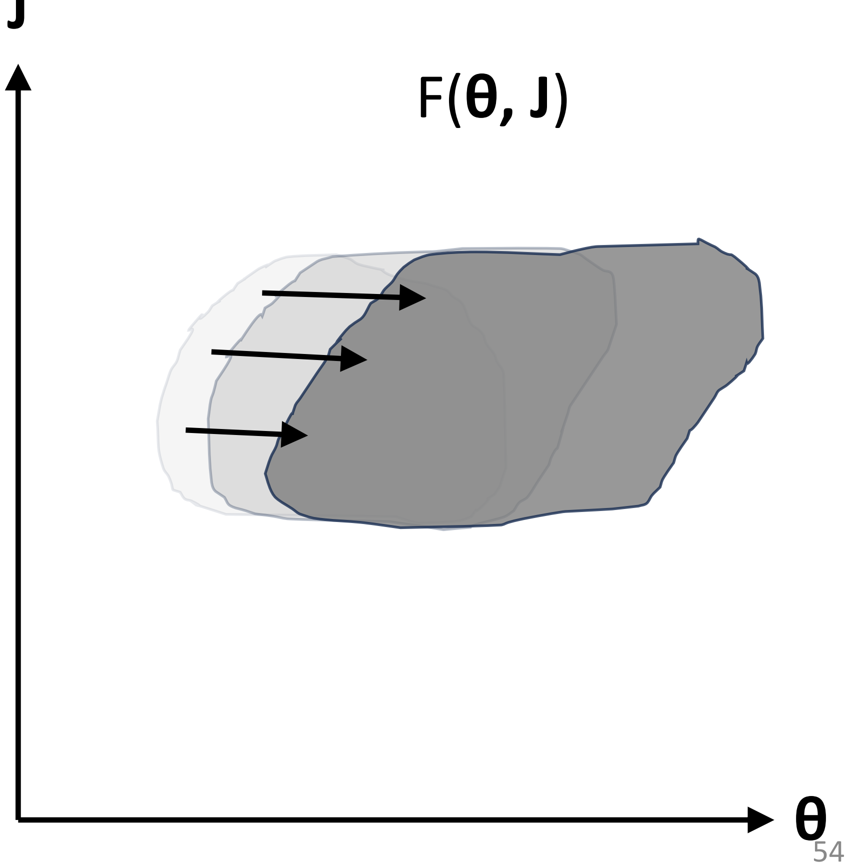
Phase mixing

$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial \boldsymbol{v}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} - \frac{\partial H}{\partial \boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = 0$$

$$\frac{\partial F}{\partial t} + \boldsymbol{\Omega}(\boldsymbol{J}) \cdot \frac{\partial F}{\partial \boldsymbol{\theta}} = 0$$

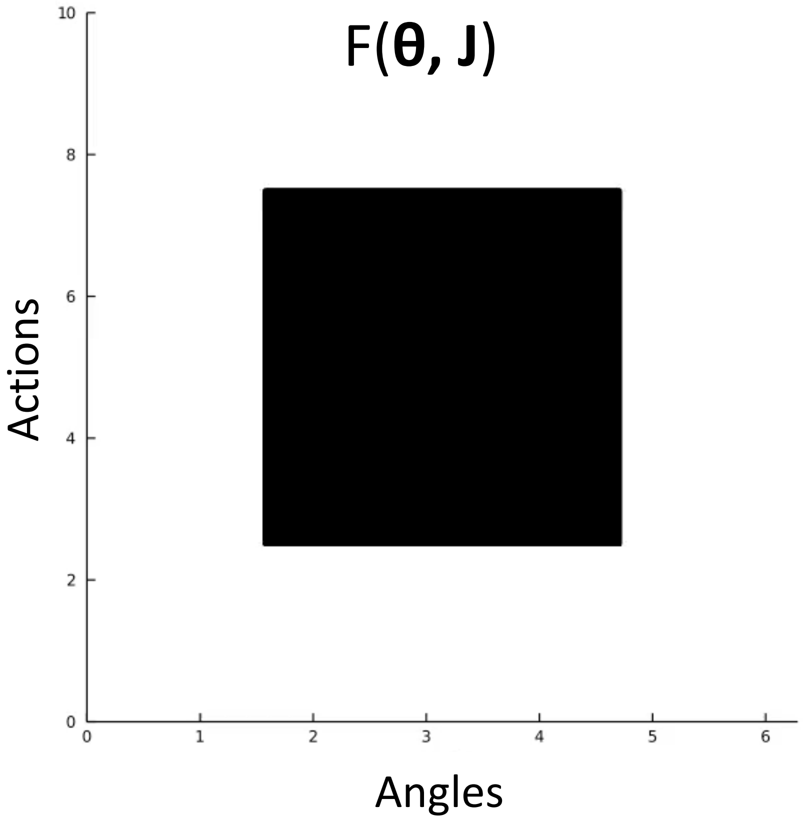


Angle-action coordinates



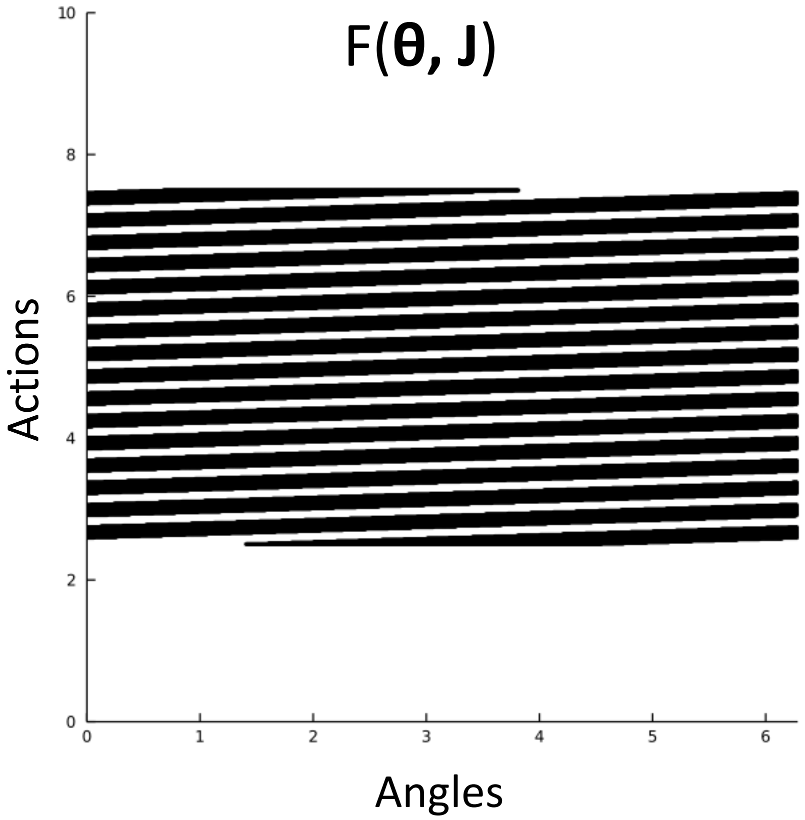
Phase mixing

- Shearing
- Phase-averaged state



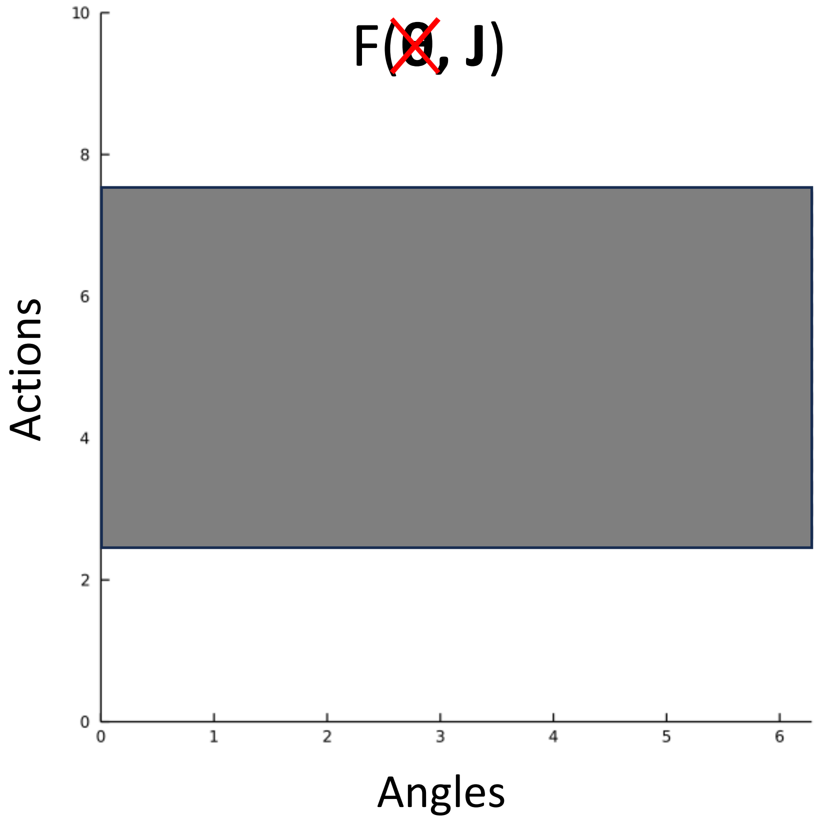
Phase mixing

- Shearing
- Phase-averaged state



blurring

$\simeq$

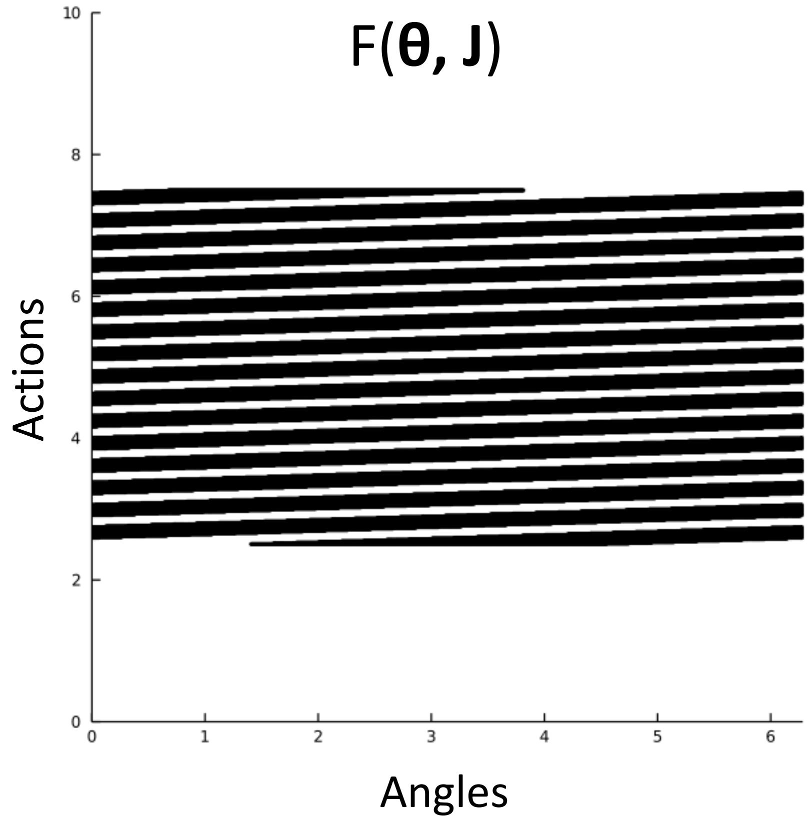


Phase mixing

- Shearing
- Phase-averaged state

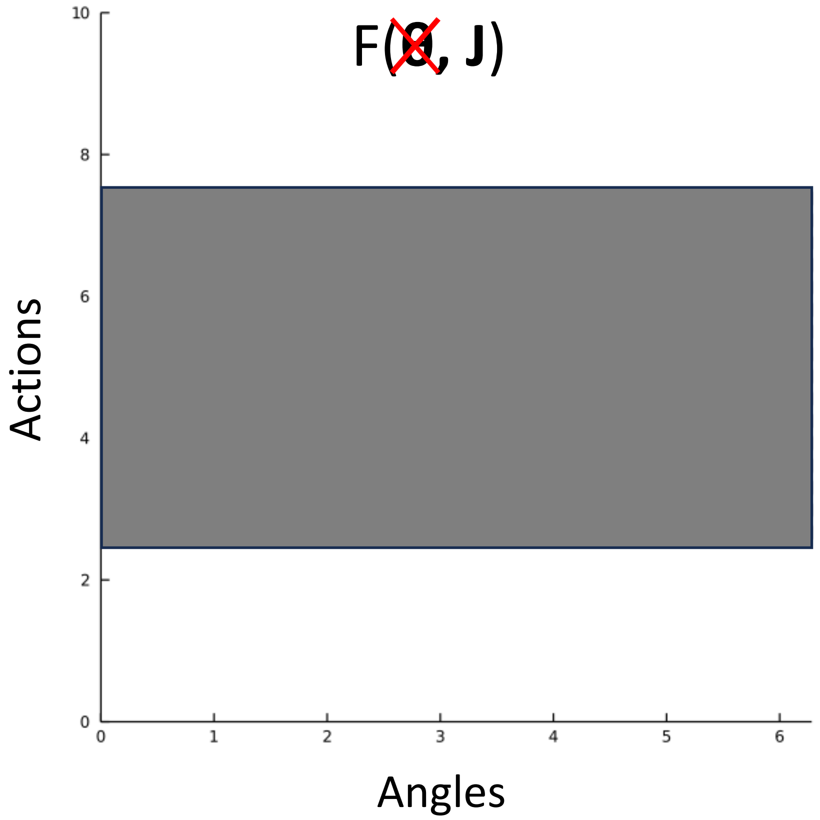
0

$$\frac{\partial F}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial F}{\partial \boldsymbol{\theta}} = 0$$

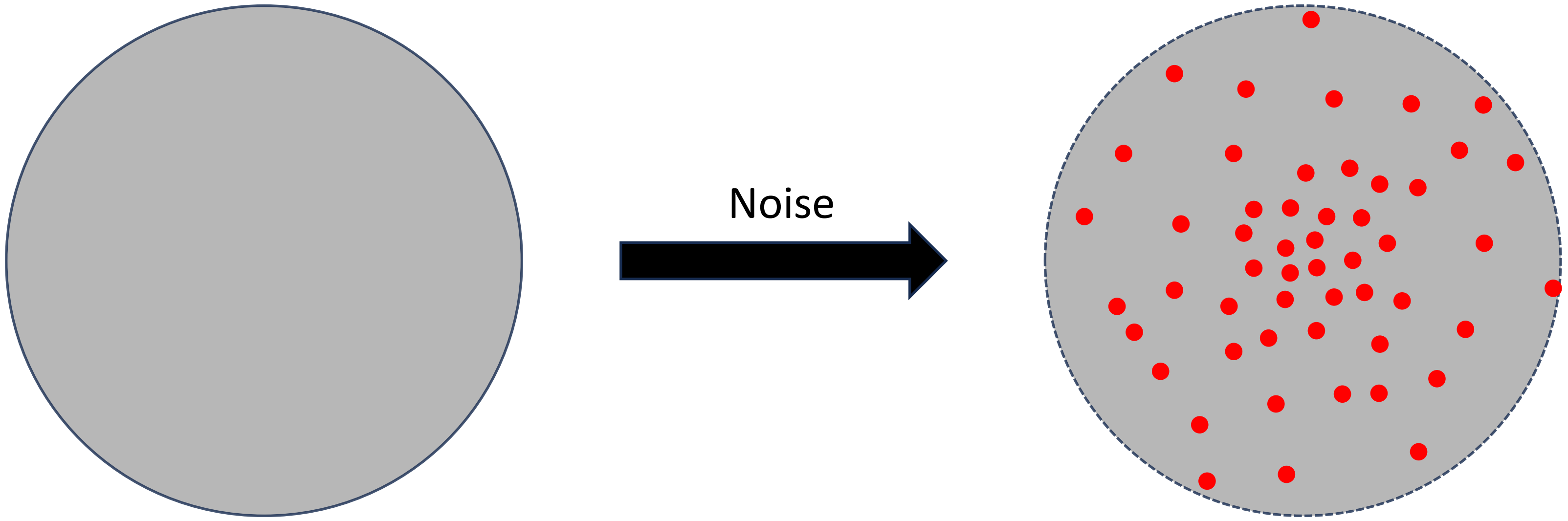


blurring

≃



Driving secular relaxation: finite-N effects



Mean-field potential

Collisionless dynamics: C[F] = 0

$$\frac{\partial F}{\partial t} + \mathbf{\Omega} \cdot \frac{\partial F}{\partial \boldsymbol{\theta}} = 0$$

QSS 0

Mean field potential + **finite-N noise**

Collisional dynamics: $\mathcal{C}[F] = \frac{1}{N}[...]$

$$\frac{\partial F}{\partial t} + \mathbf{\Omega} \cdot \frac{\partial F}{\partial \boldsymbol{\theta}} = \mathcal{C}[F]$$

QSS 0

Computing the collision integral C[F]

- **How to make theoretical predictions ?**
- What mechanisms impact secular evolution?
- How does kinematics impact evolution ?

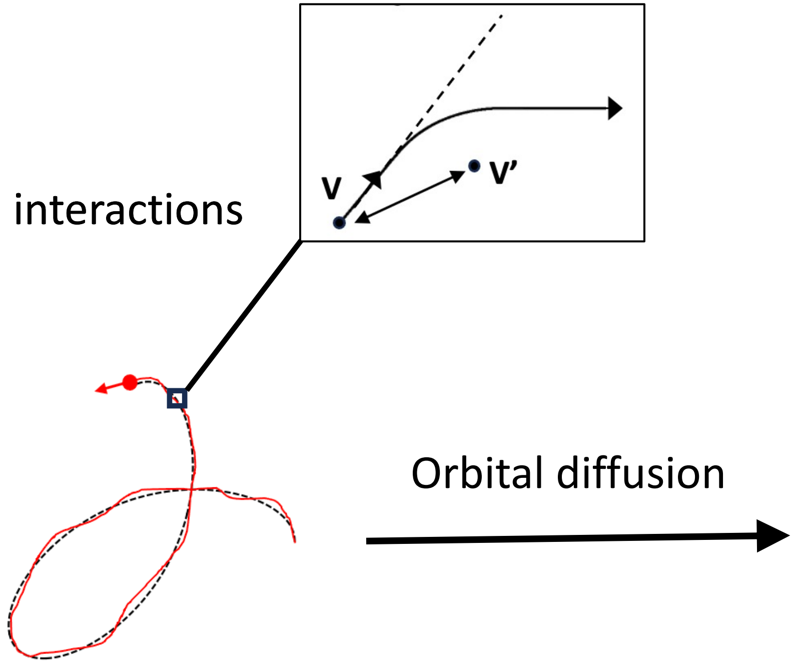
Orbital diffusion

$$\frac{\partial F}{\partial t}(\boldsymbol{J}, t) = \mathcal{C}[F]$$

Orbital diffusion

$$\frac{\partial F}{\partial t}(\boldsymbol{J}, t) = \boxed{\mathcal{C}[F]}$$

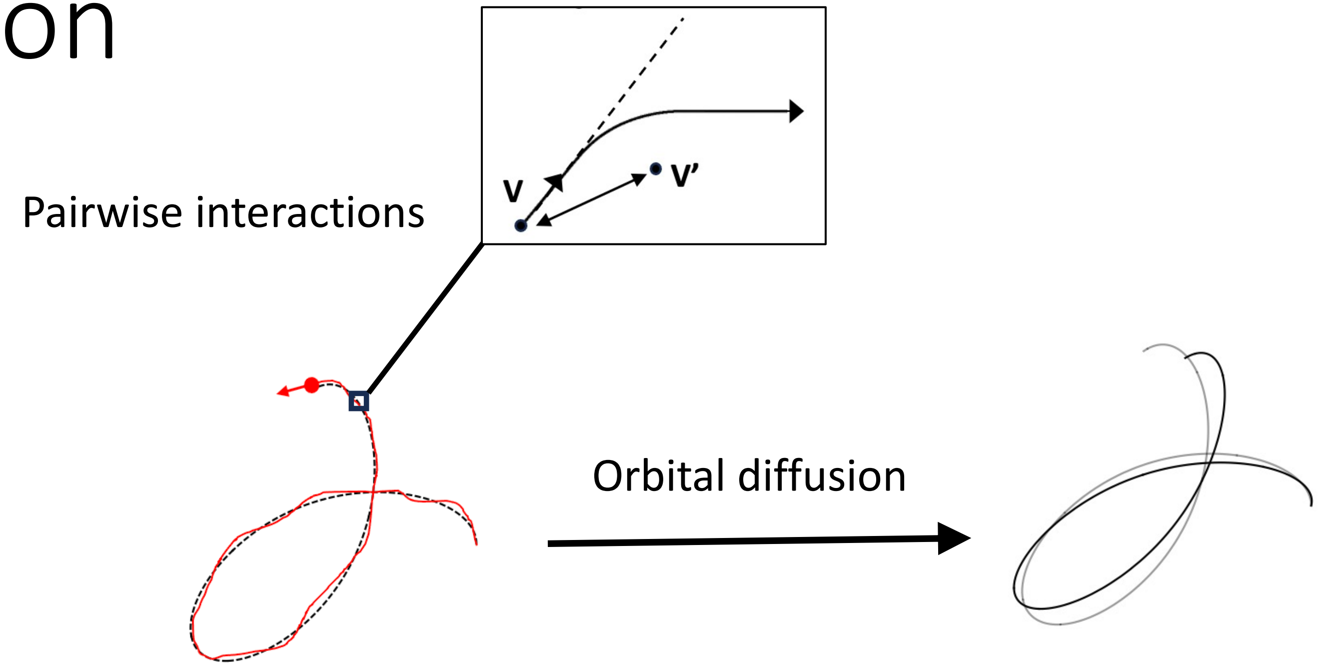
Pairwise interactions



Orbital diffusion

Orbital diffusion

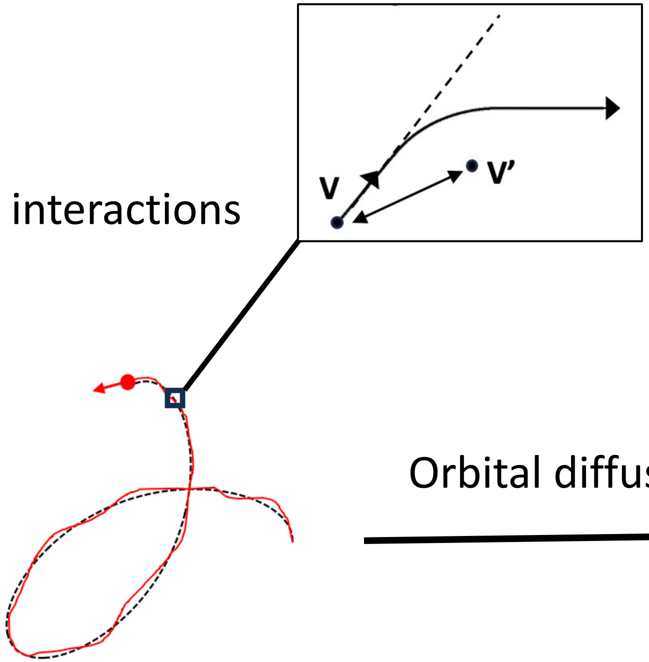
$$\frac{\partial F}{\partial t}(\boldsymbol{J}, t) = \boxed{\mathcal{C}[F]}$$

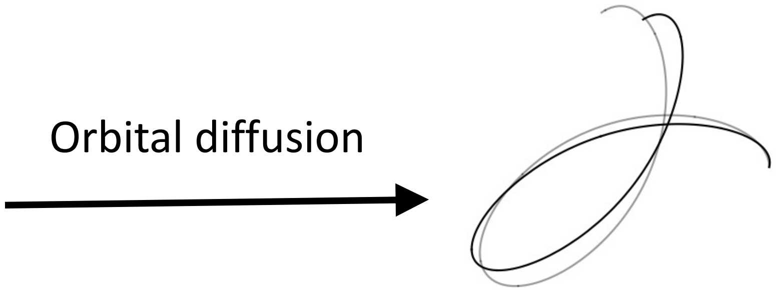


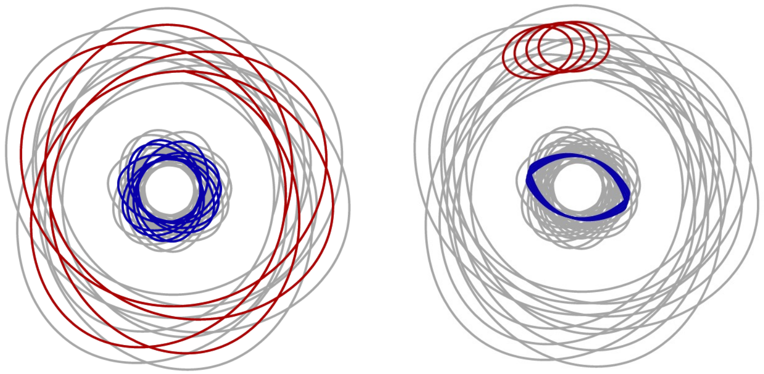
Orbital diffusion

Pairwise interactions

$$\frac{\partial F}{\partial t}(\boldsymbol{J}, t) = \boxed{\mathcal{C}[F]}$$

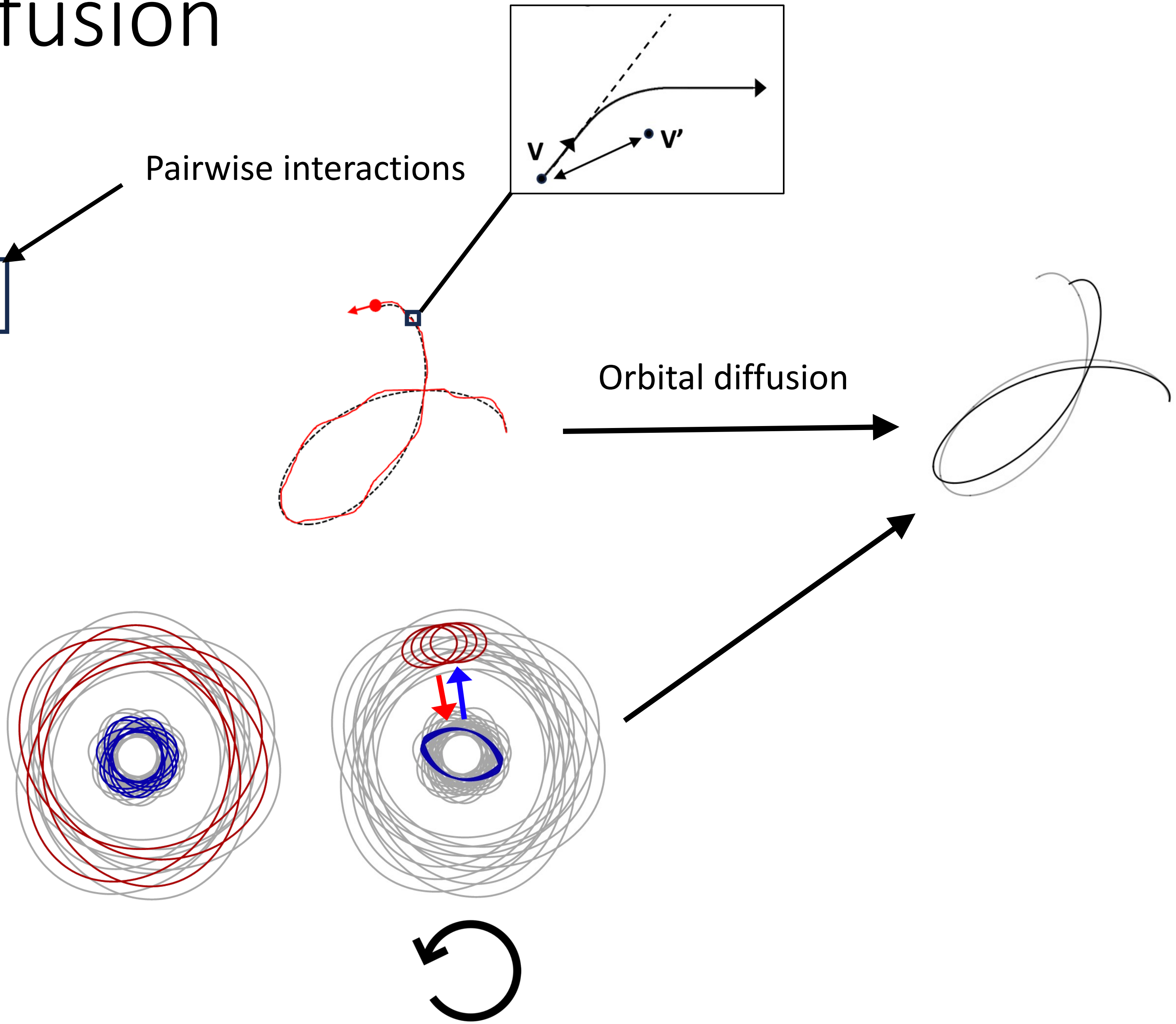






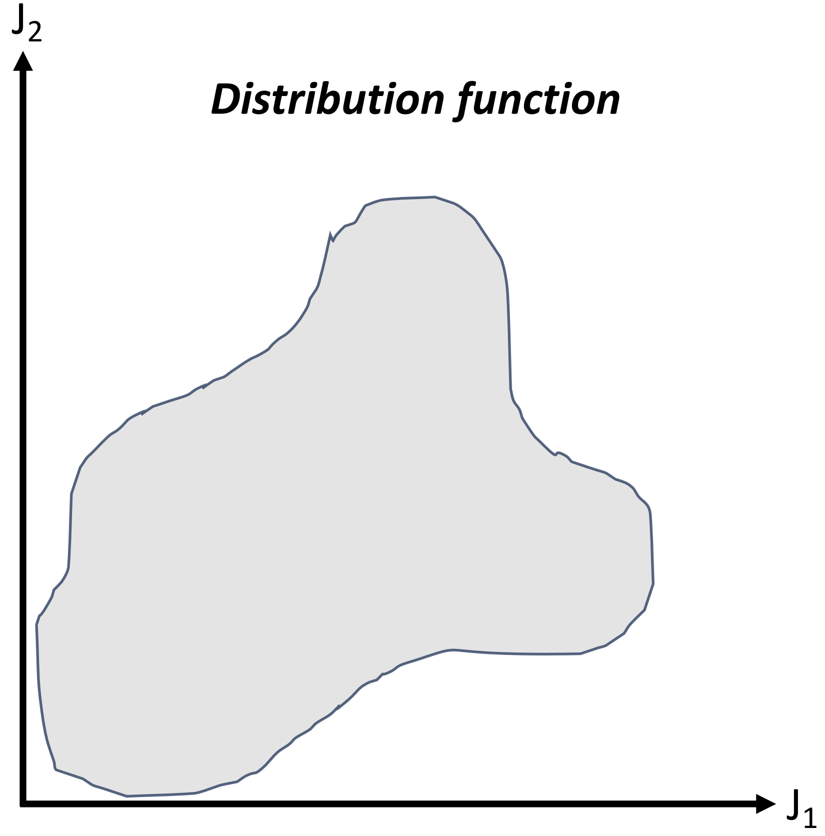
Orbital diffusion

$$\frac{\partial F}{\partial t}(\boldsymbol{J}, t) = \boxed{\mathcal{C}[F]}$$

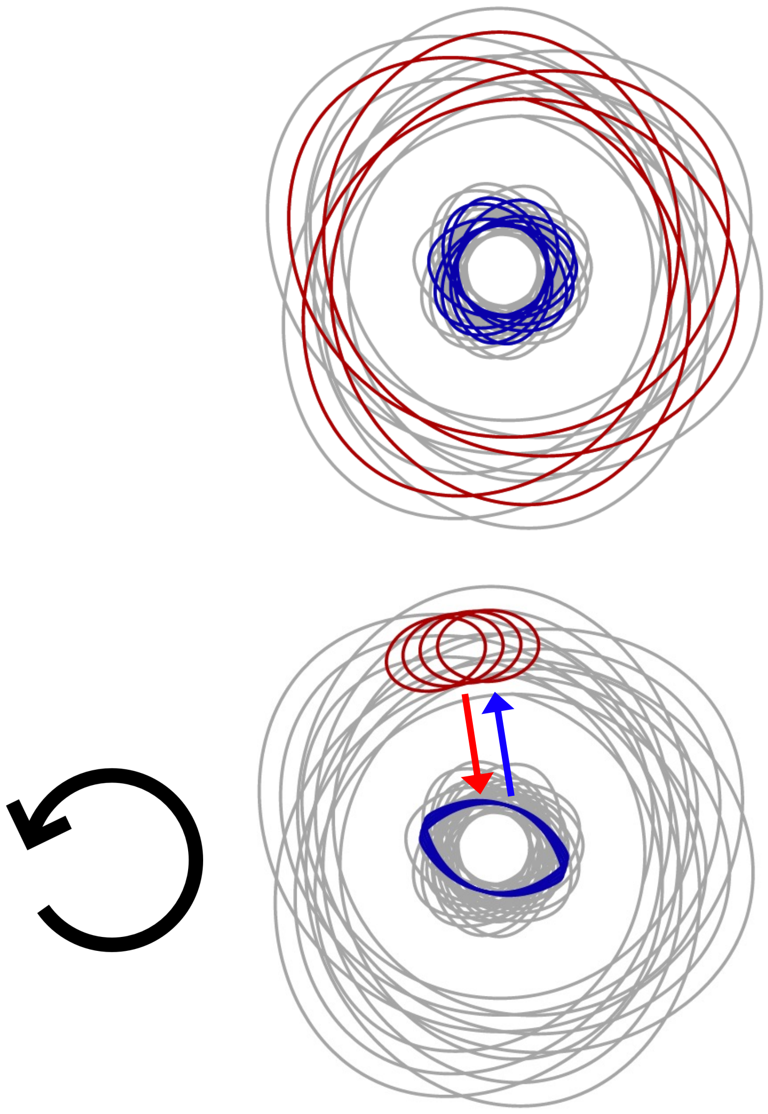


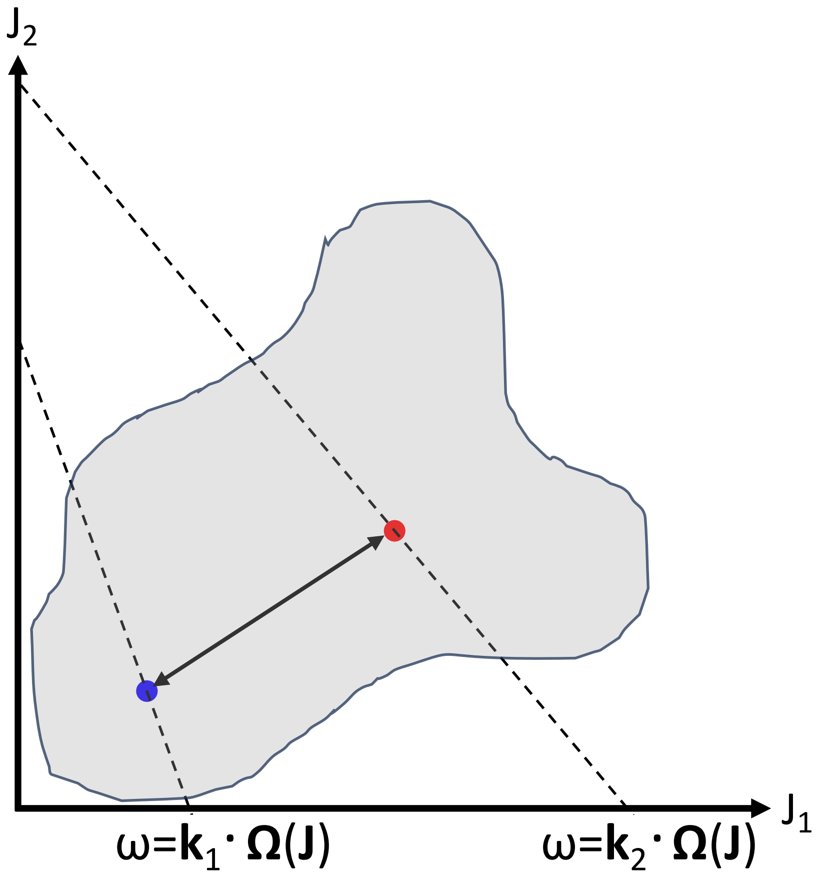
Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J}, t) = \mathcal{C}[F]$$

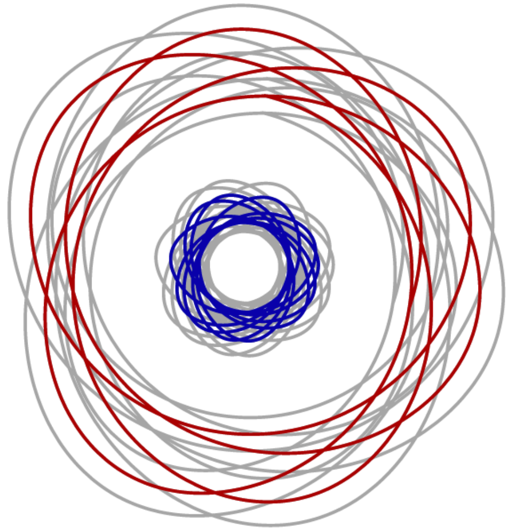


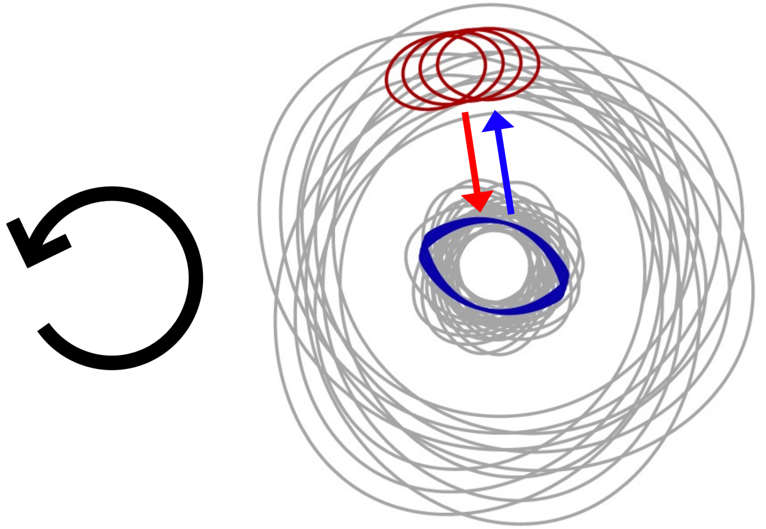
Balescu-Lenard equation

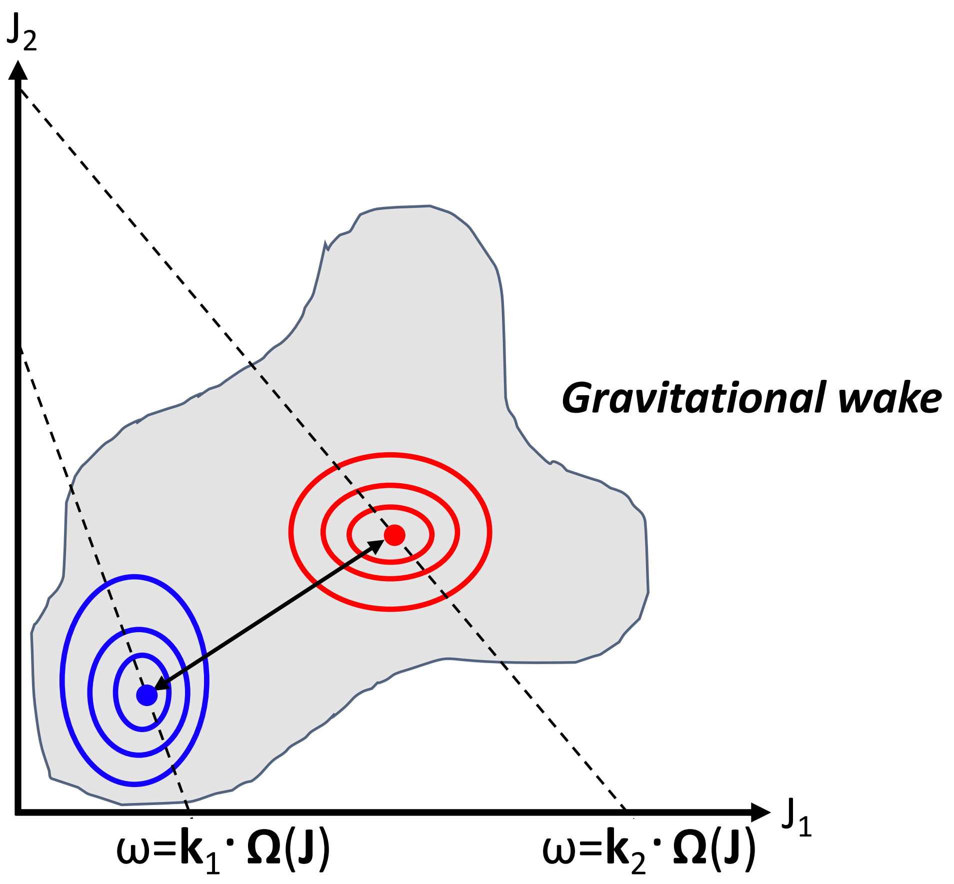




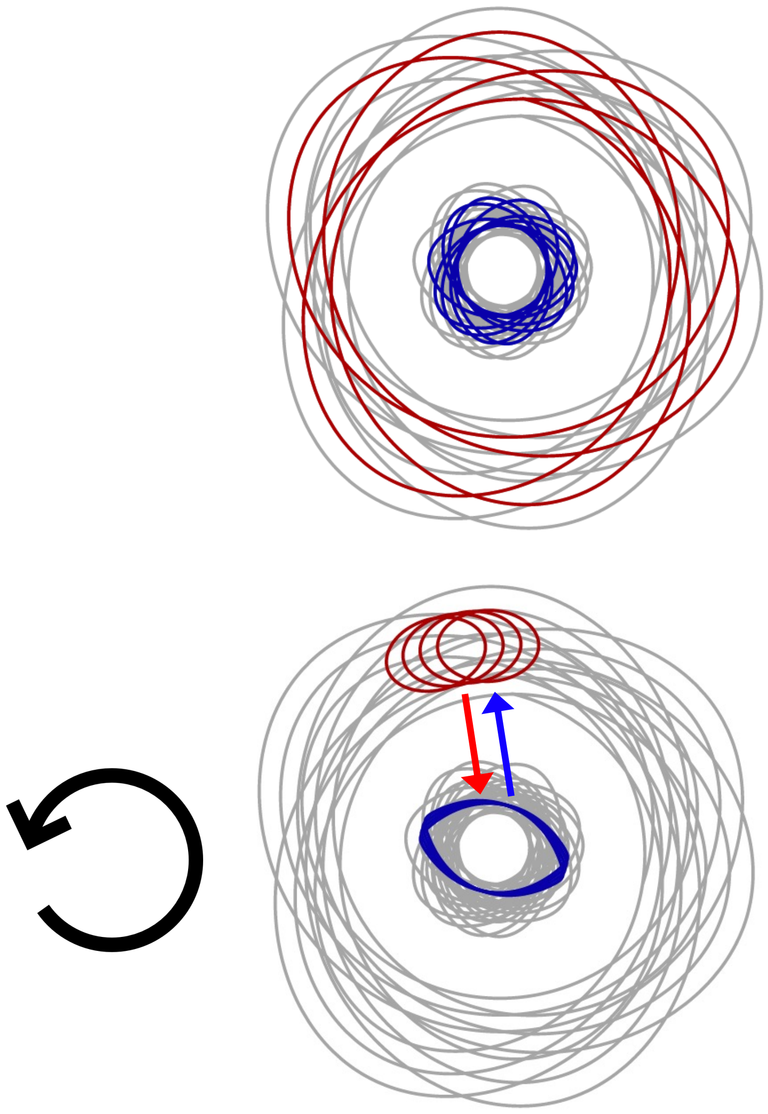
Balescu-Lenard equation

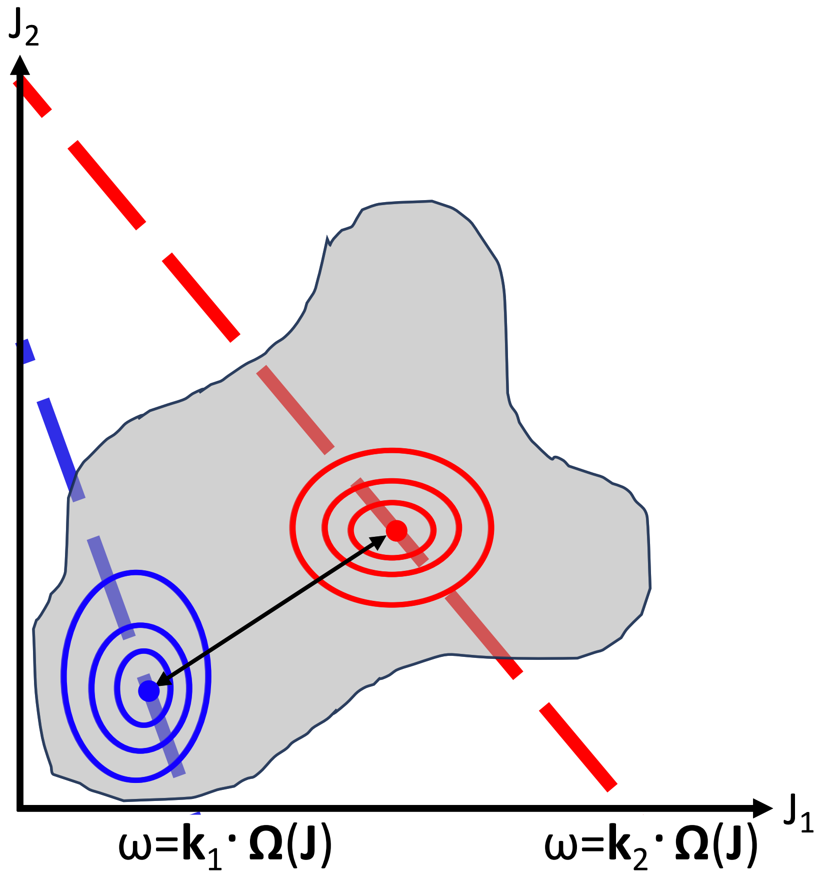






Balescu-Lenard equation





Heyvaerts (2010)
Chavanis (2012)

Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J},t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' |\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \, \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')$$
$$\times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J},t) F(\boldsymbol{J}',t),$$

Heyvaerts (2010)
Chavanis (2012)

Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J},t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' |\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \, \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}') \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J},t)F(\boldsymbol{J}',t),$$

$F(\boldsymbol{J},t)$ Slow evolution of QSS

Heyvaerts (2010)
Chavanis (2012)

Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J},t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' |\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \, \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}') \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J},t) F(\boldsymbol{J}',t),$$

$\frac{M}{N}$ Shot noise fluctuations

$F(\boldsymbol{J},t)$ Slow evolution of QSS

Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J},t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' |\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \, \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}') \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J},t)F(\boldsymbol{J}',t),$$

$\frac{M}{N}$ Shot noise fluctuations

$F(\boldsymbol{J},t)$ Slow evolution of QSS

$\sum_{\boldsymbol{k},\boldsymbol{k}'}$ Sum over resonances

Heyvaerts (2010)
Chavanis (2012)

Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J}, t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' |\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}') \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J}, t) F(\boldsymbol{J}', t),$$

$\frac{M}{N}$ Shot noise fluctuations

$\delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')$ Non-local resonant coupling

$F(\boldsymbol{J}, t)$ Slow evolution of QSS

$\sum_{\boldsymbol{k},\boldsymbol{k}'}$ Sum over resonances

Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J},t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' |\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}') \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J},t)F(\boldsymbol{J}',t),$$

$\frac{M}{N}$	Shot noise fluctuations	$\delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')$	Non-local resonant coupling
$F(\boldsymbol{J},t)$	Slow evolution of QSS	$\|\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})\|^2$	Dressed orbital coupling
$\sum_{\boldsymbol{k},\boldsymbol{k}'}$	Sum over resonances		

Heyvaerts (2010)
Chavanis (2012)

Balescu-Lenard equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J},t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' \, |\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \, \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}') \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J},t) F(\boldsymbol{J}',t),$$

$\frac{M}{N}$	Shot noise fluctuations	$\delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')$	Non-local resonant coupling
$F(\boldsymbol{J},t)$	Slow evolution of QSS	$\|\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})\|^2$	Dressed orbital coupling
$\sum_{\boldsymbol{k},\boldsymbol{k}'}$	Sum over resonances	$\int \mathrm{d}\boldsymbol{J}'$	Scan over action space

Limit cases of the BL equation

Heyvaerts (2010)

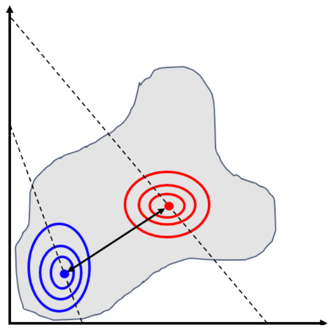
Balescu-Lenard
(BL)

$$\sum_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})=\boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}')$$

$$\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}$$

$$\int \mathrm{d}\boldsymbol{J}'$$



Limit cases of the BL equation

Heyvaerts (2010)

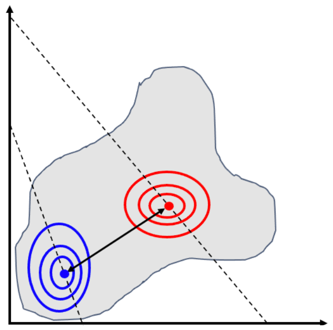
Polyachenko & Shukhman (1982)
Chavanis (2012)

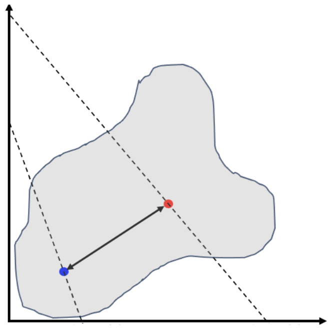
Balescu-Lenard (BL) → **No self-gravity** → Landau (RR)

$$\sum_{\boldsymbol{k},\boldsymbol{k}'} \quad \boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})=\boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}') \quad \psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'} \quad \int \mathrm{d}\boldsymbol{J}'$$

No self-amplification →

$$\sum_{\boldsymbol{k},\boldsymbol{k}'} \quad \boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})=\boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}') \quad \psi_{\boldsymbol{k}\boldsymbol{k}'} \quad \int \mathrm{d}\boldsymbol{J}'$$

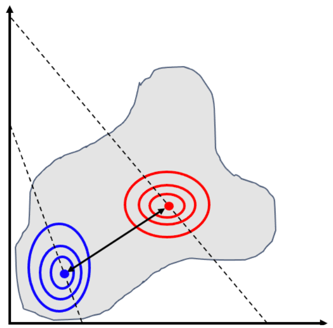


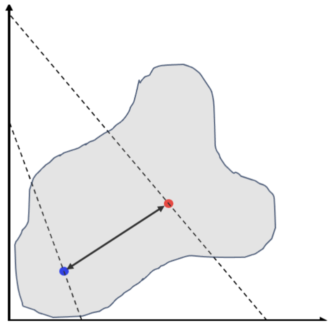


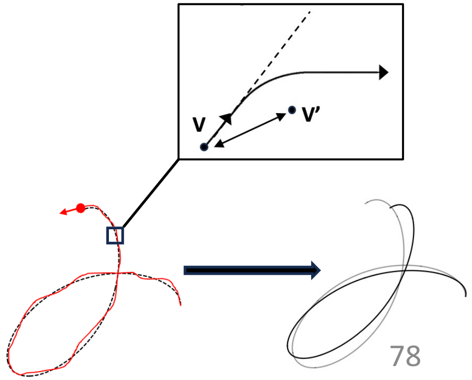
Limit cases of the BL equation

Heyvaerts (2010)		*Polyachenko & Shukhman (1982)* *Chavanis (2012)*		*Chandrasekhar (1943)* *Chavanis (2013)*
Balescu-Lenard (BL)	**No self-gravity** →	Landau (RR)	**Local homogeneity** →	Orbit-averaged Chandrasekhar
$\sum_{\boldsymbol{k},\boldsymbol{k}'}$		$\sum_{\boldsymbol{k},\boldsymbol{k}'}$	→	$\int \mathrm{d}\boldsymbol{k}$
$\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})=\boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}')$		$\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})=\boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}')$		$\boldsymbol{k}\cdot\boldsymbol{v}=\boldsymbol{k}\cdot\boldsymbol{v}'$
$\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}$		$\psi_{\boldsymbol{k}\boldsymbol{k}'}$	→	$\widehat{u}(\mathbf{k})$
$\int \mathrm{d}\boldsymbol{J}'$		$\int \mathrm{d}\boldsymbol{J}'$	→	$\int \mathrm{d}\boldsymbol{v}'$

Local deflections

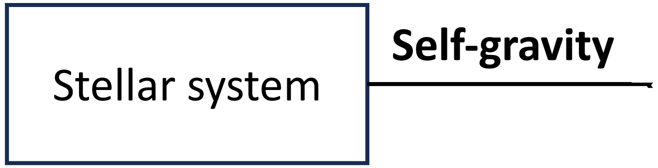




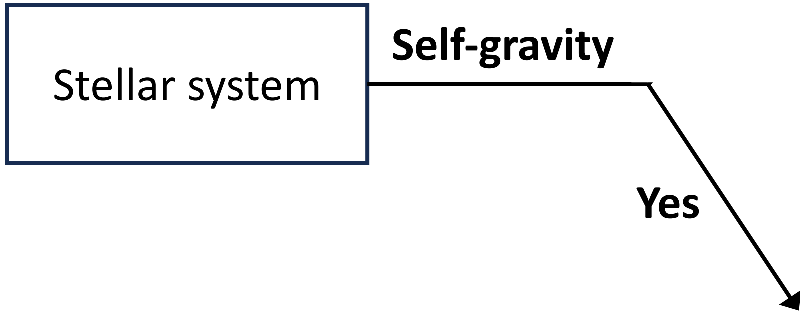


Possible applications

Possible applications



Possible applications



Possible applications

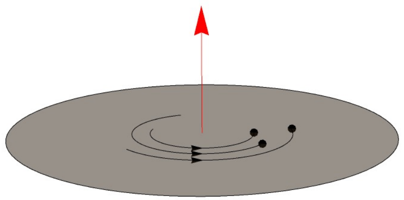
Stellar system

Self-gravity

Yes

Balescu-Lenard theory

Cold system:
galactic disc



Possible applications

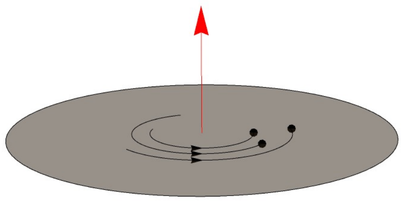
Stellar system

Self-gravity

Yes

Balescu-Lenard theory

Cold system: galactic disc



Bar formation

Reddish, …, Tep & al. (2022)

Possible applications

Stellar system

Self-gravity

No

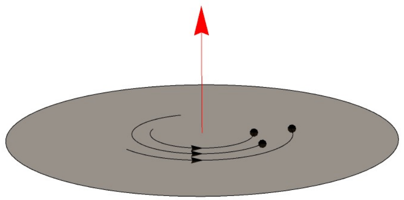
Landau theory

Hot system

Yes

Balescu-Lenard theory

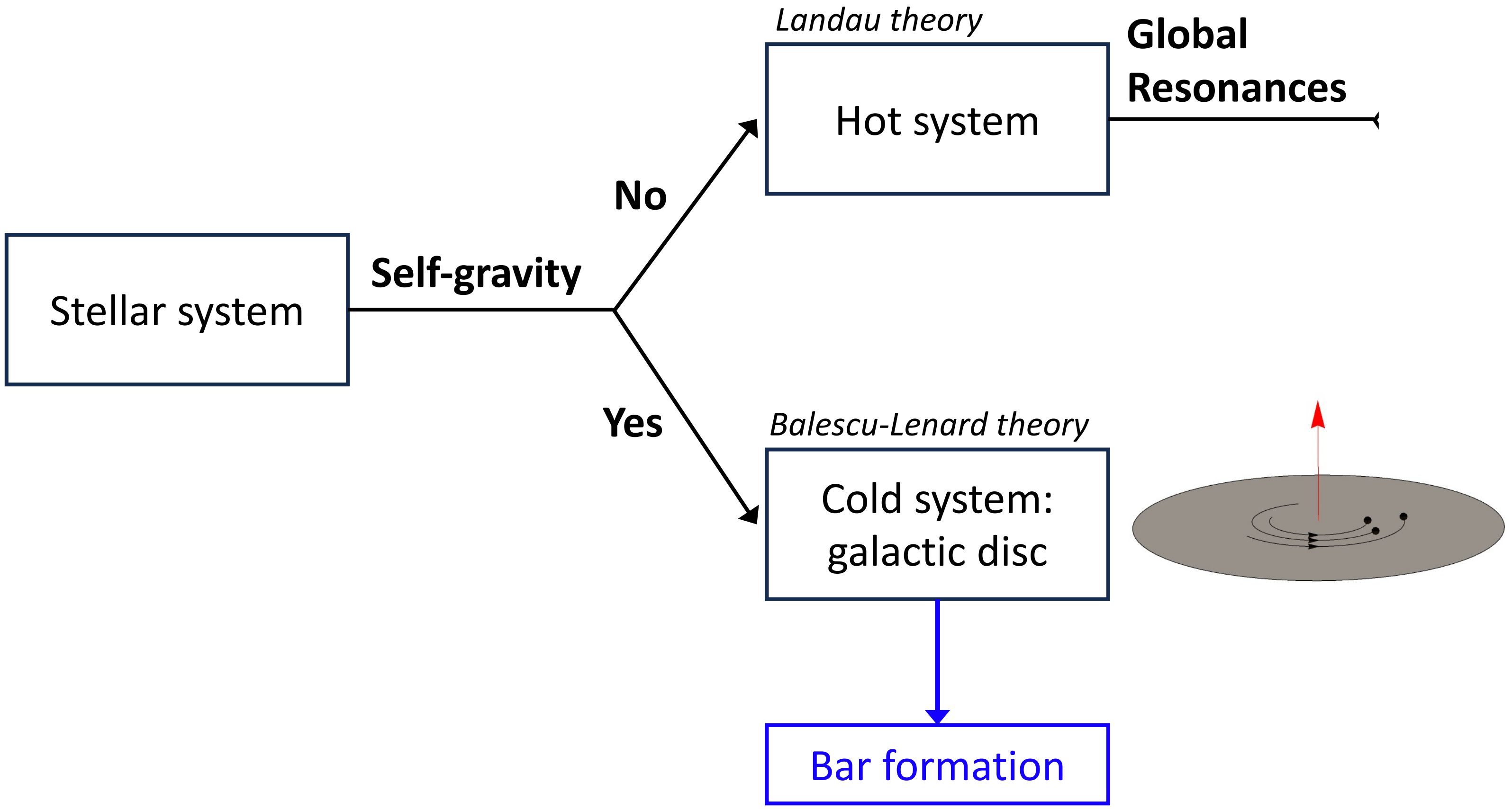
Cold system: galactic disc



Bar formation

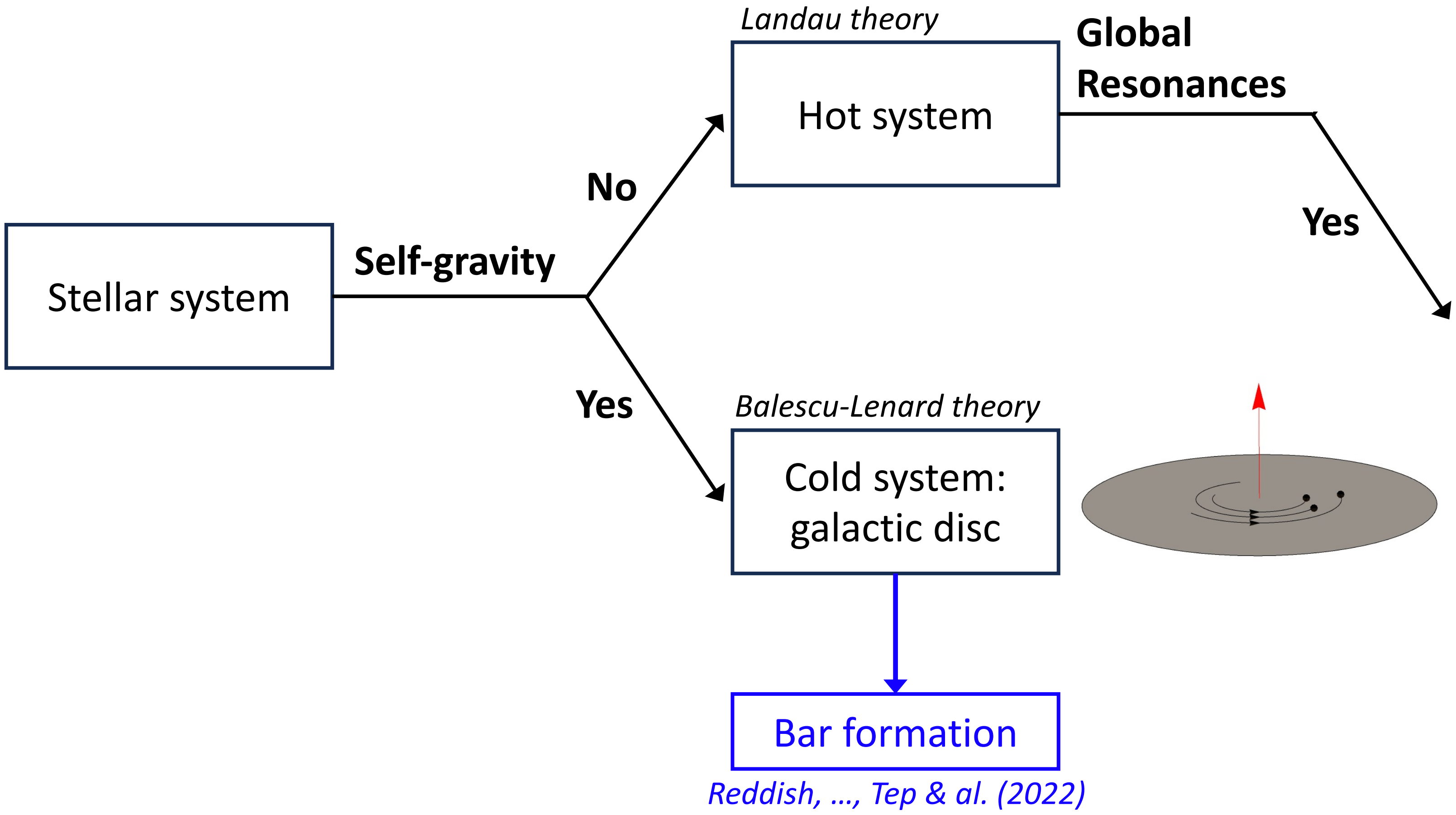
Reddish, …, Tep & al. (2022)

Possible applications

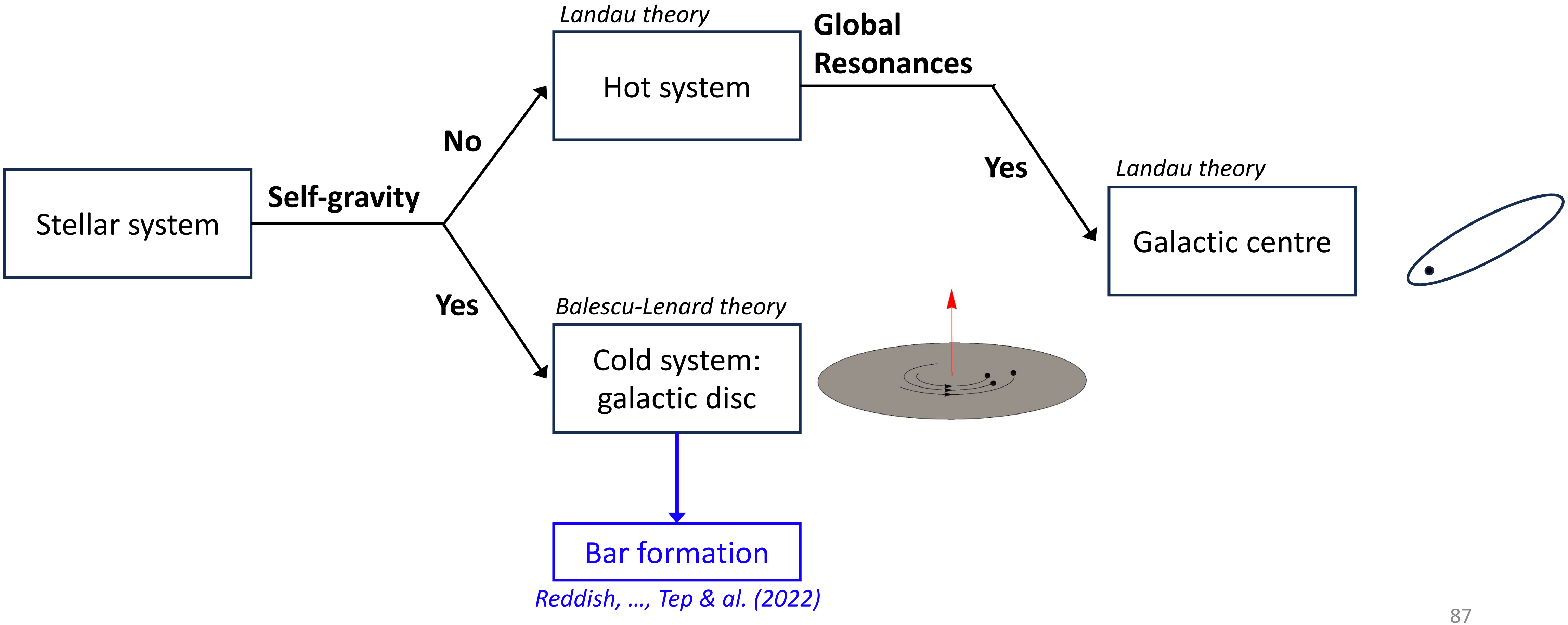


Reddish, …, Tep & al. (2022)

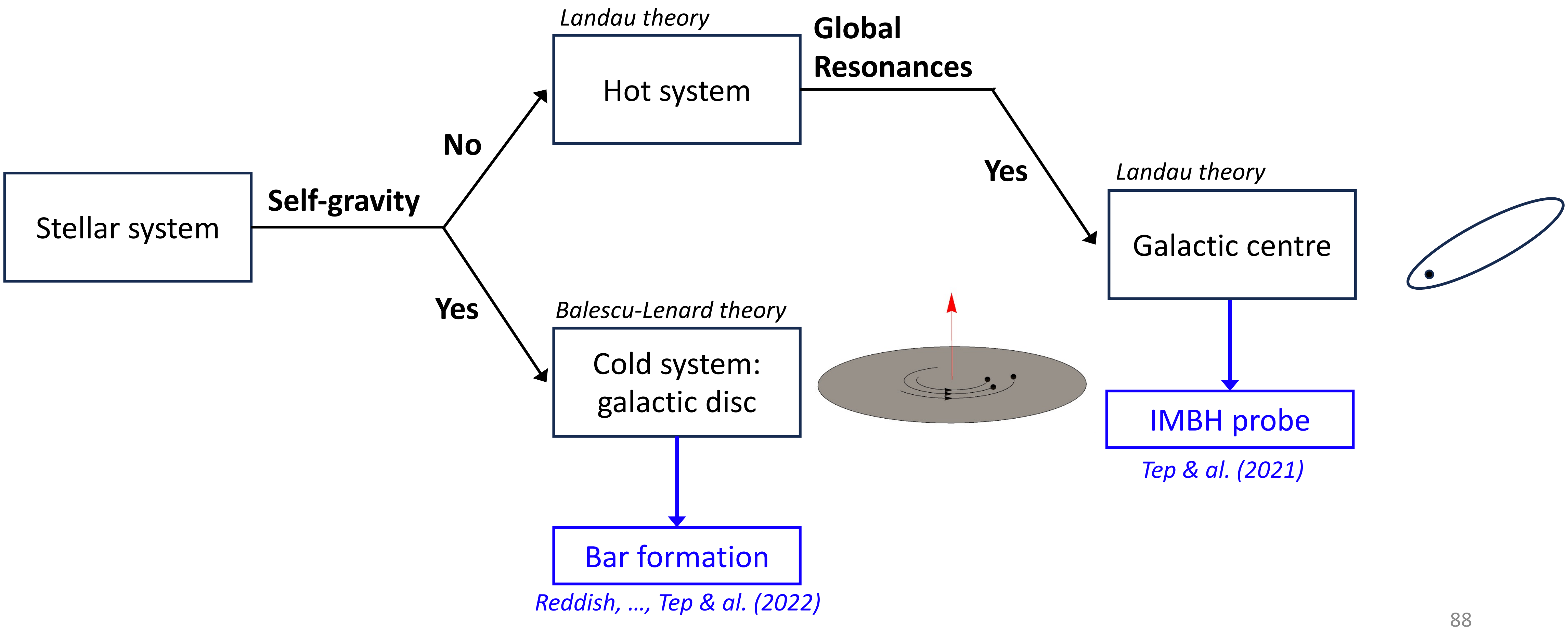
Possible applications



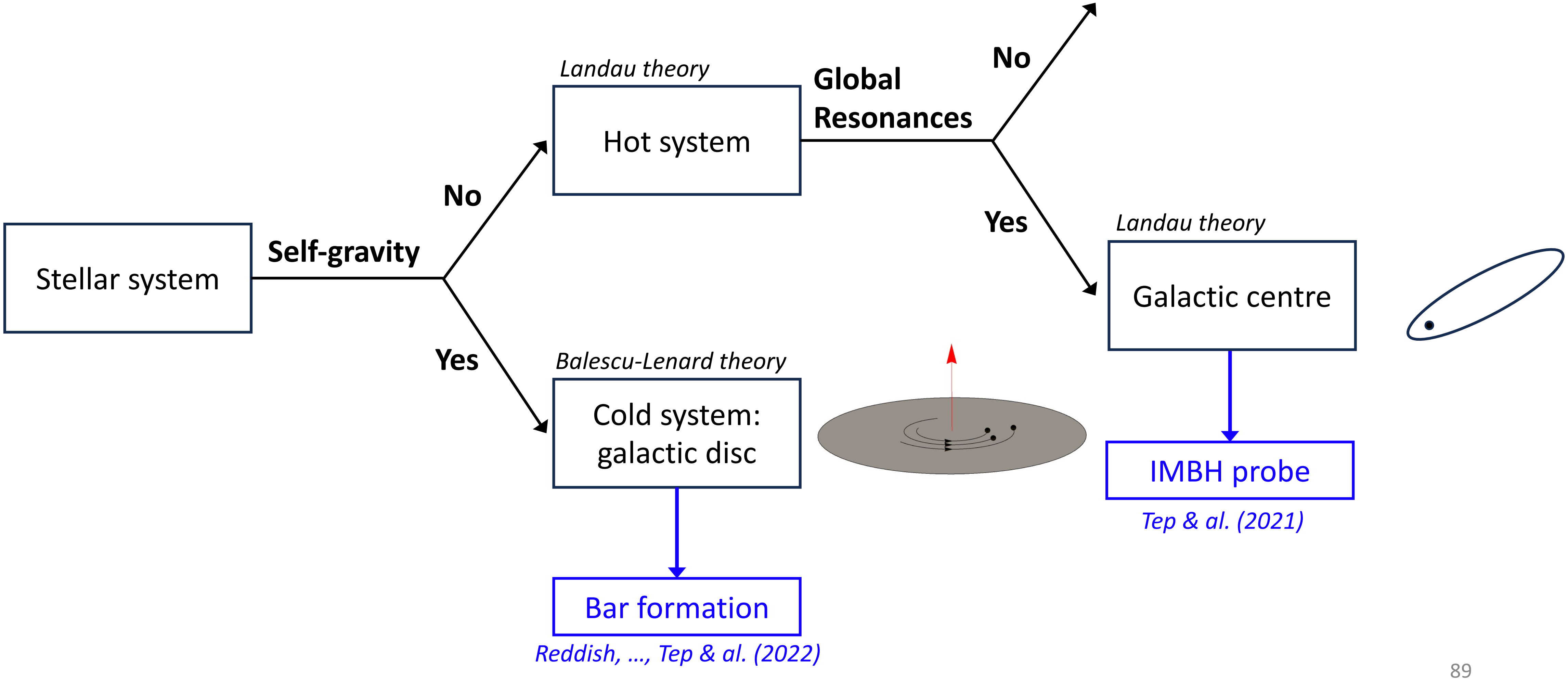
Possible applications



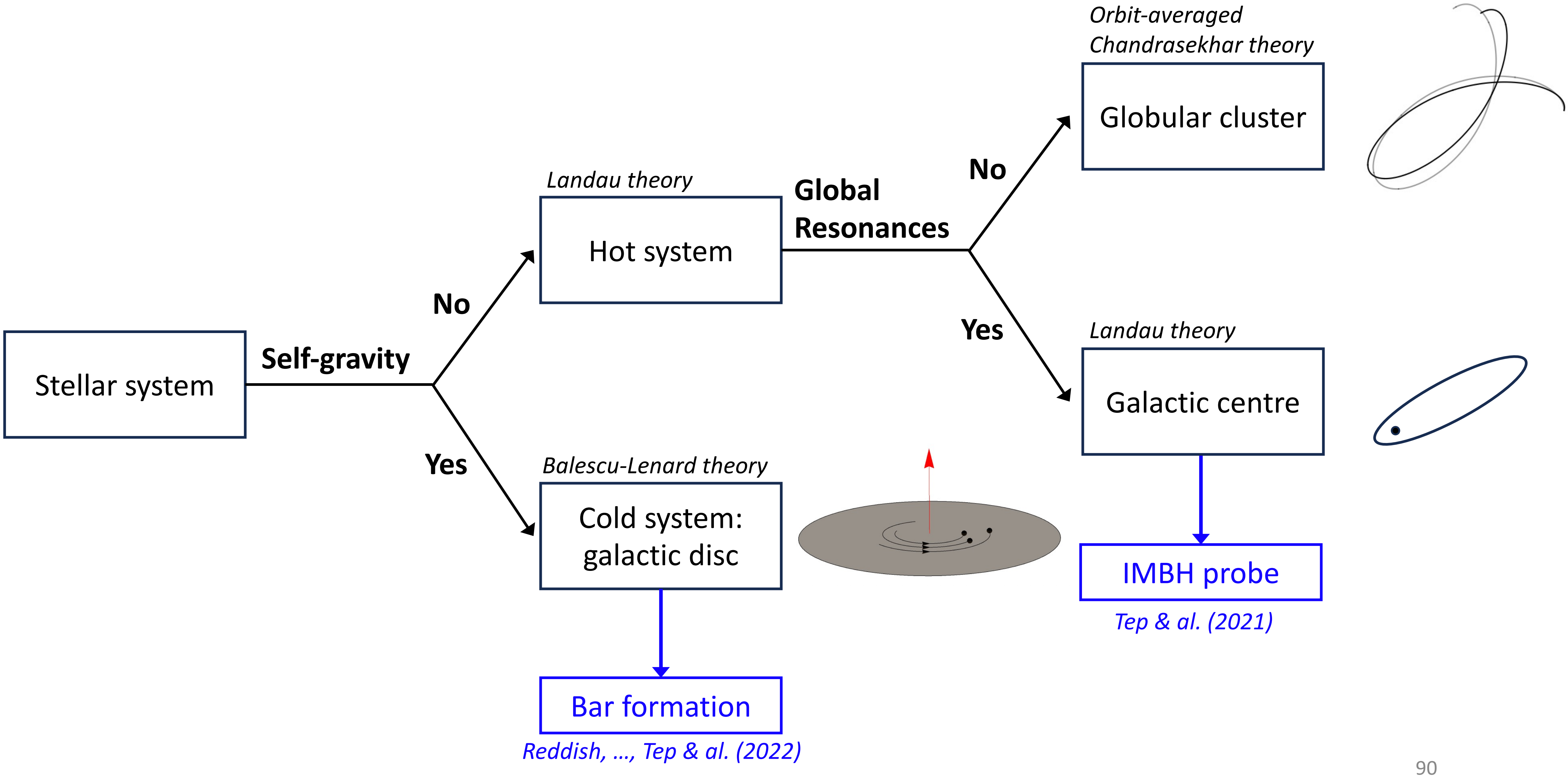
Possible applications



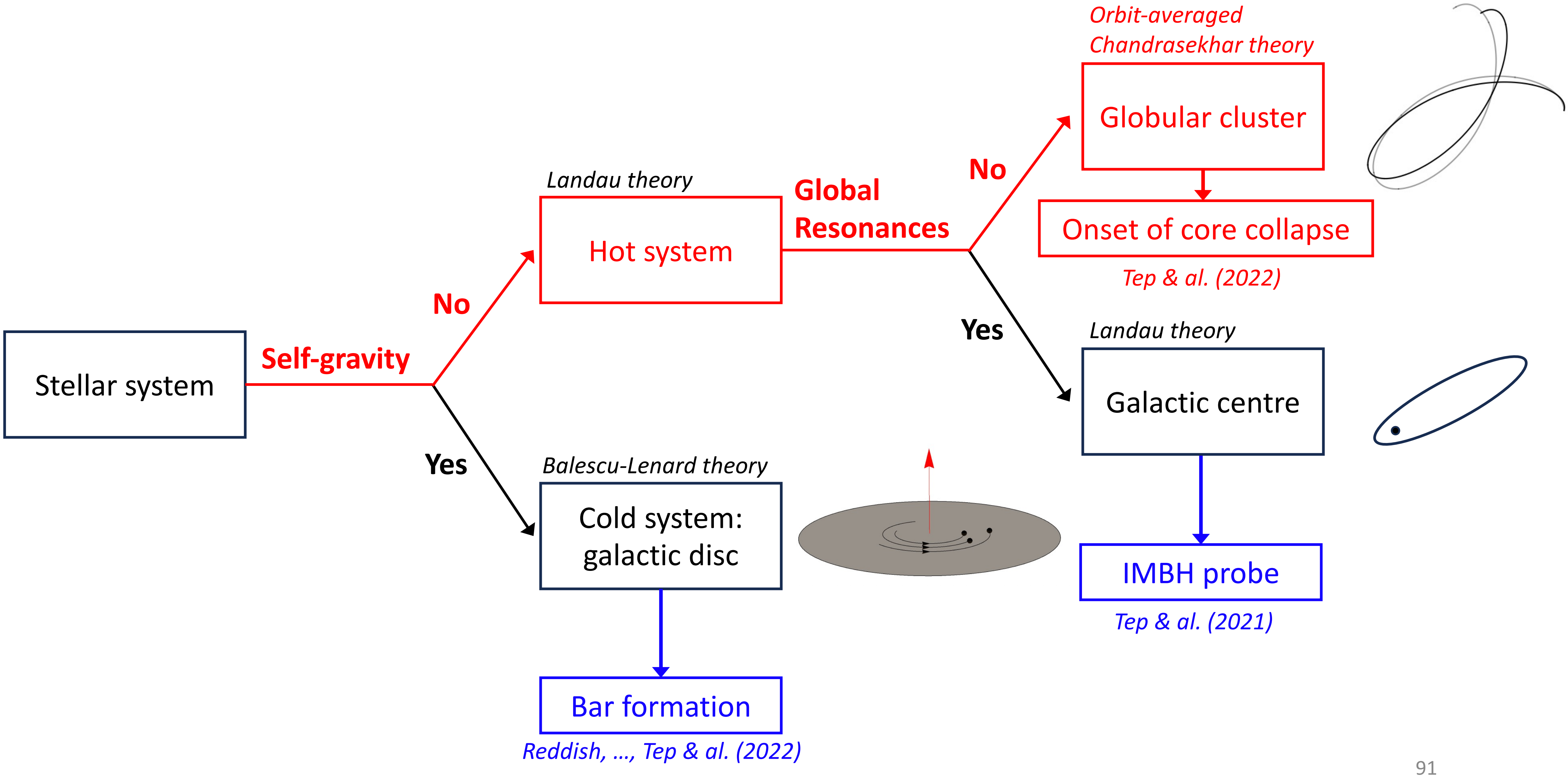
Possible applications



Possible applications



Possible applications



Secular predictions

Credit: *ESA/Hubble & NASA,R.Cohen*

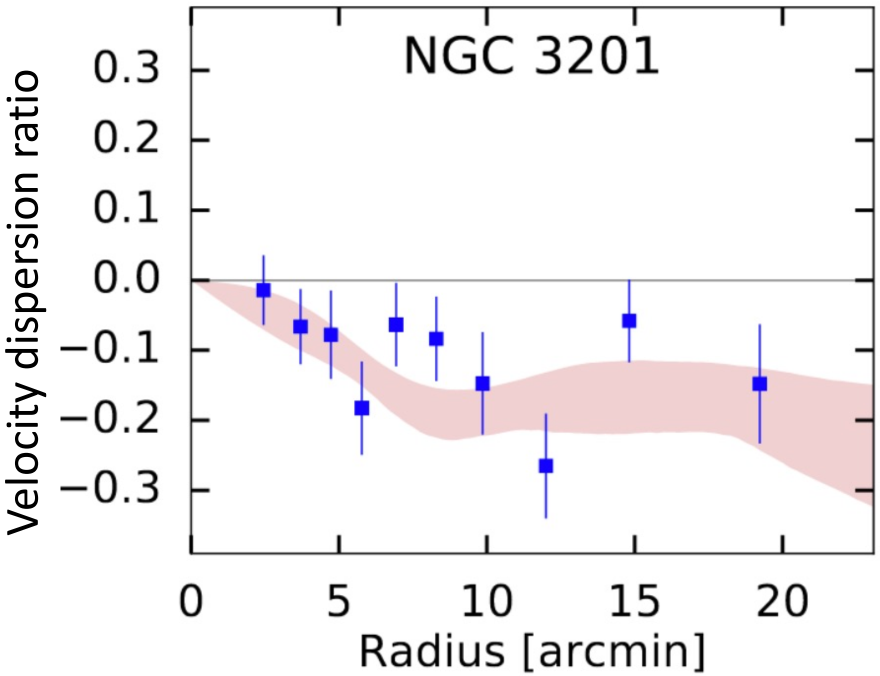
NGC 6638 (HST)

- How to make theoretical predictions ?
- **What mechanisms impact secular evolution?**
- How does kinematics impact evolution ?

Globular clusters: observations

- GAIA data: globular clusters can be anisotropic

Radially anisotropic



Vasiliev (2021)

Globular clusters: observations

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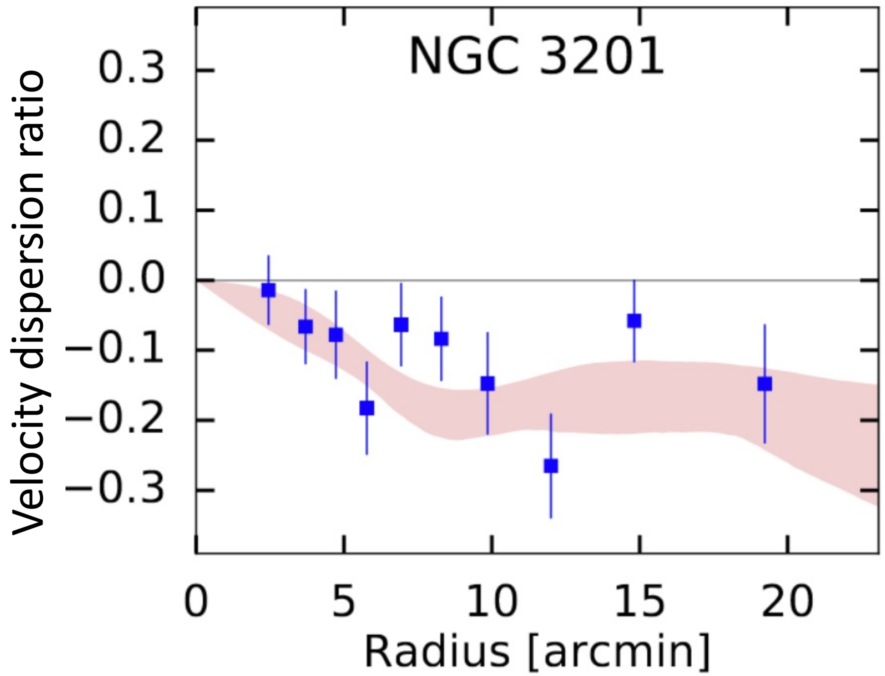
Radially anisotropic



Tangential dispersion < radial dispersion

→ **Radial anisotropy**

Vasiliev (2021)

Globular clusters: observations

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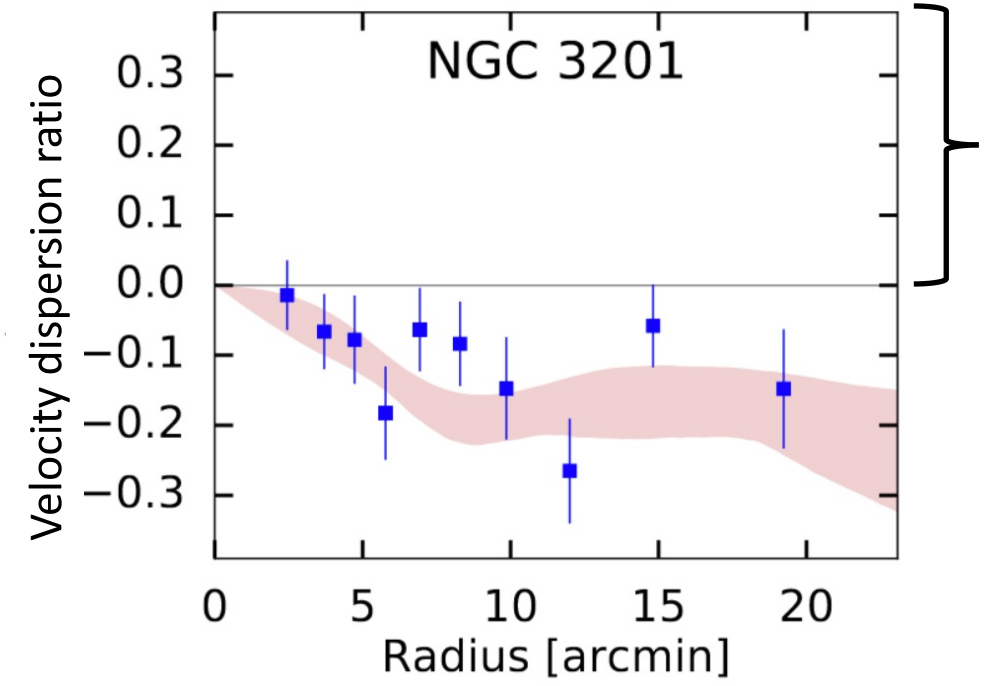
Radially anisotropic



Tangential dispersion > radial dispersion
→ **Tangential anisotropy**

Vasiliev (2021)

Globular clusters: observations

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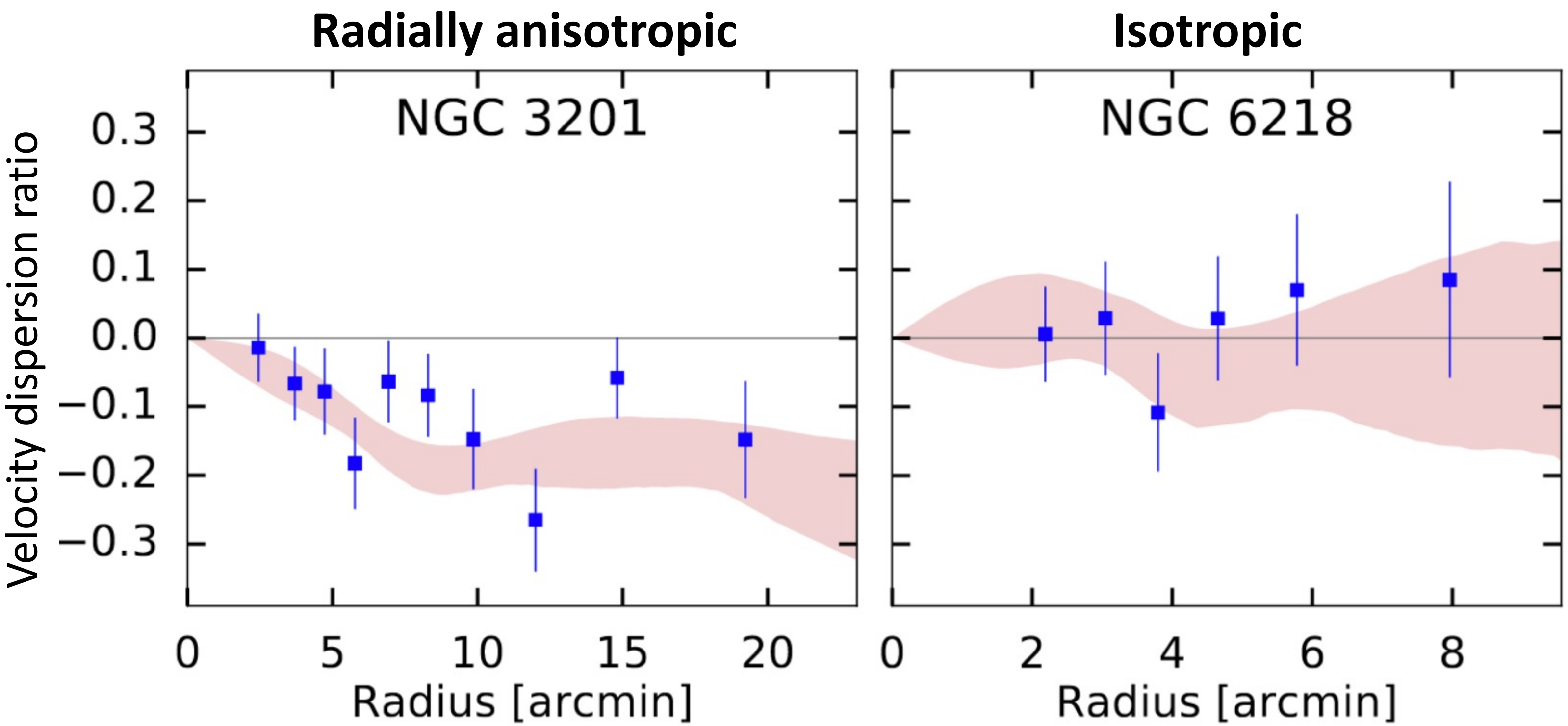

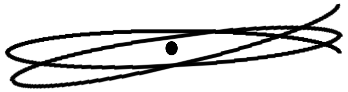

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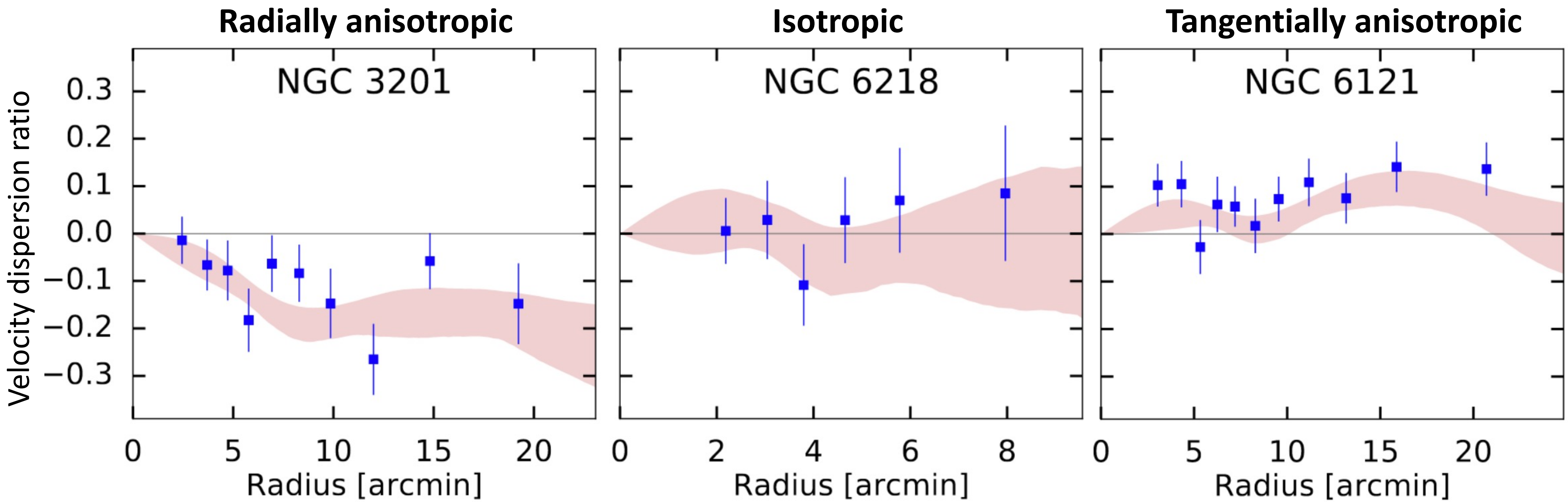
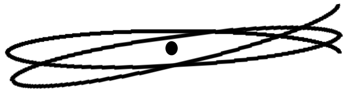
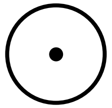


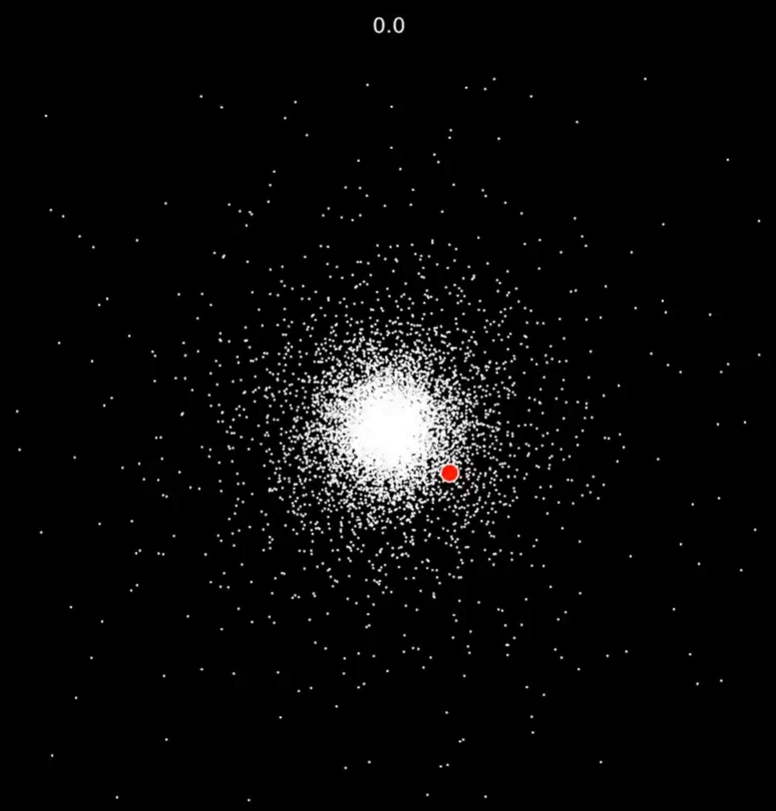




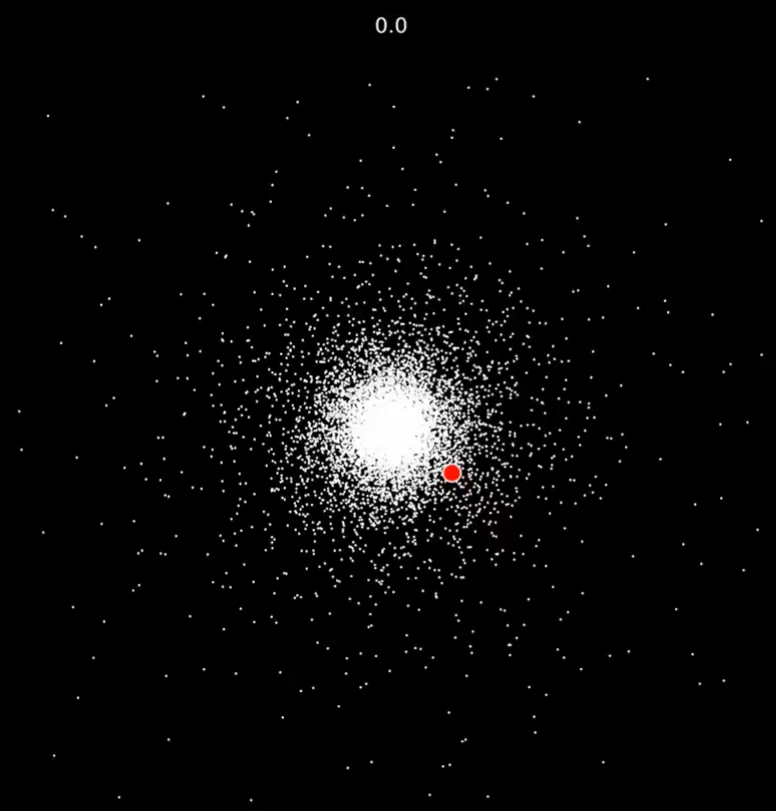
Vasiliev (2021)

The Plummer cluster (N-body simulations)

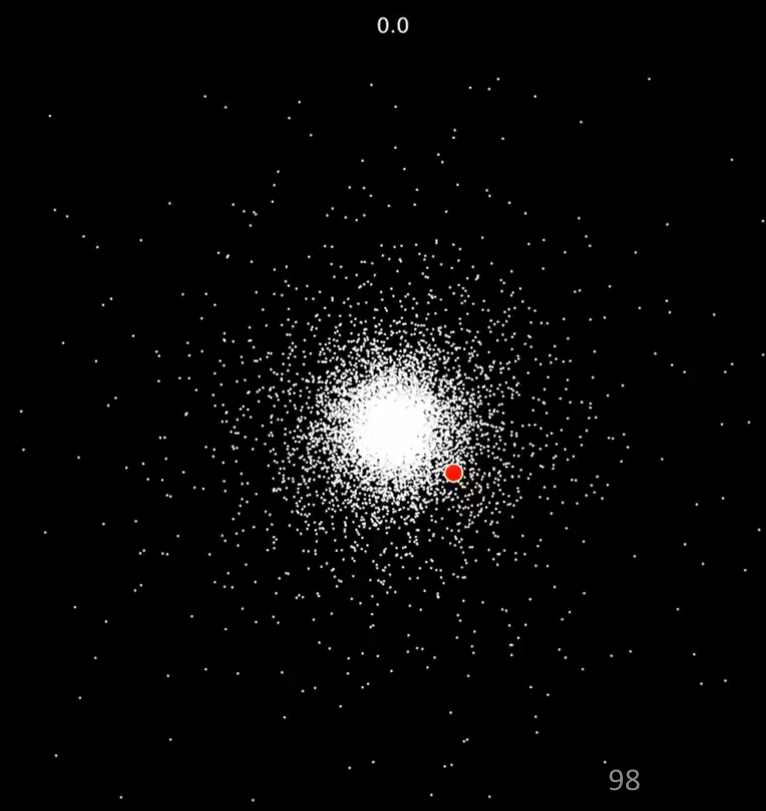
Radial anisotropy



Isotropy

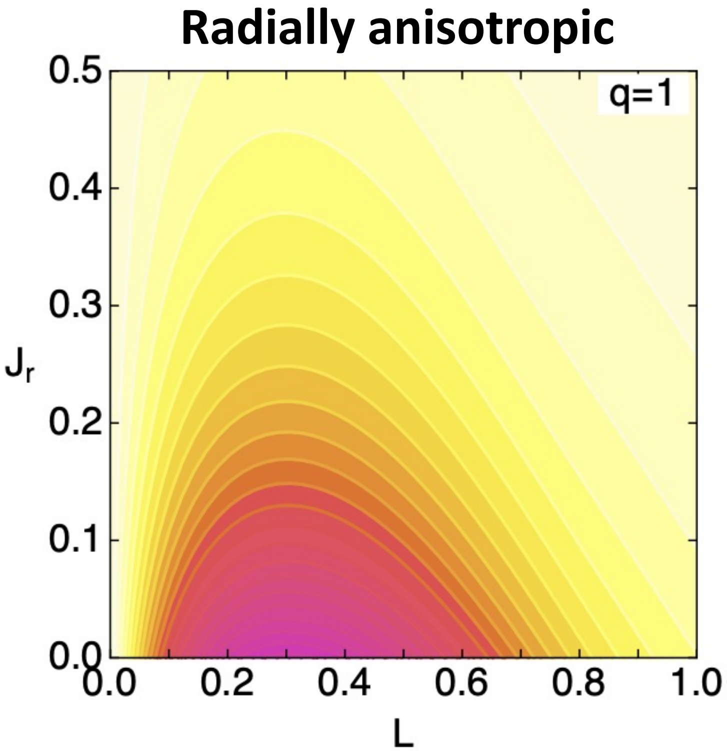


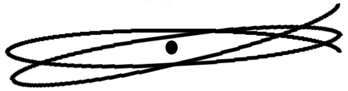
Tangential anisotropy

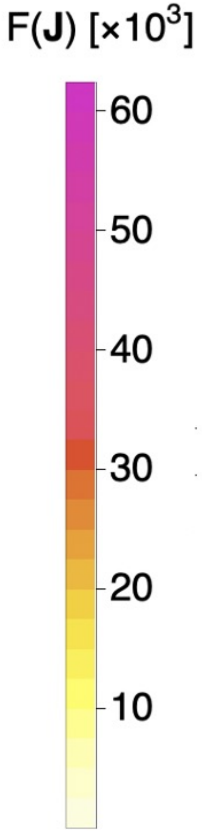


The Plummer cluster (distribution function)

- Choice of mean field: Plummer potential
- Choice of velocity dispersion: anisotropy







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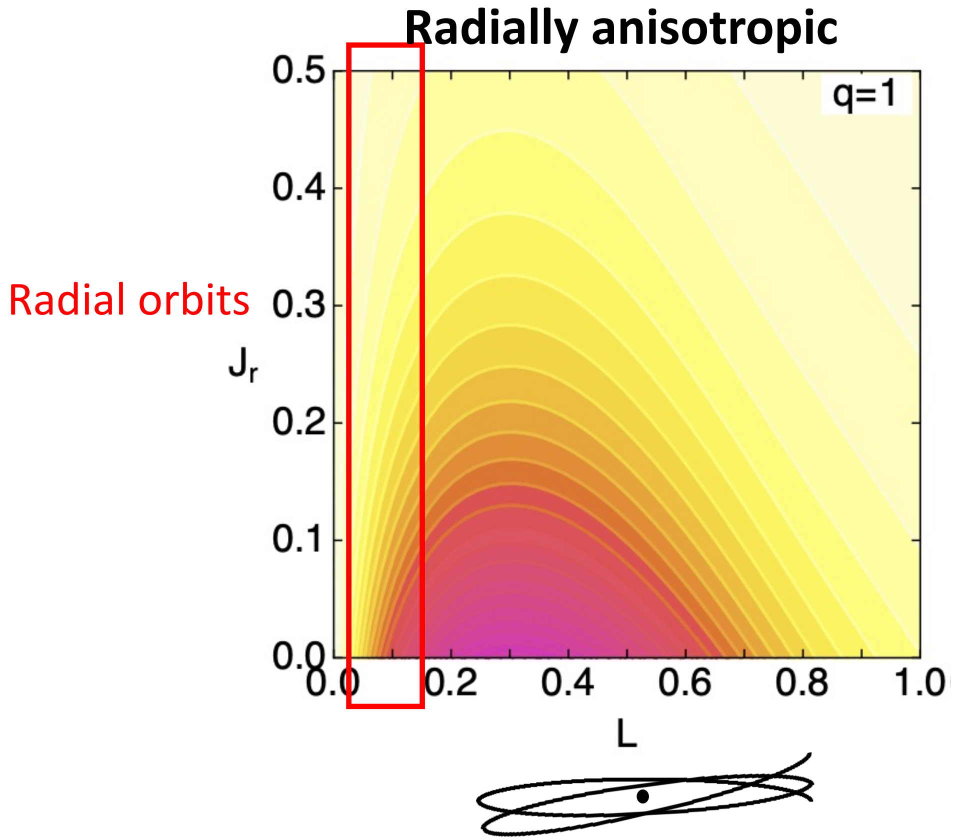

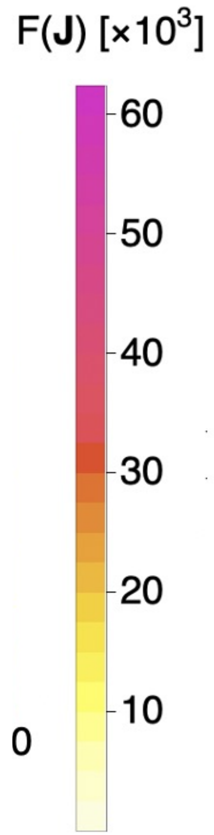

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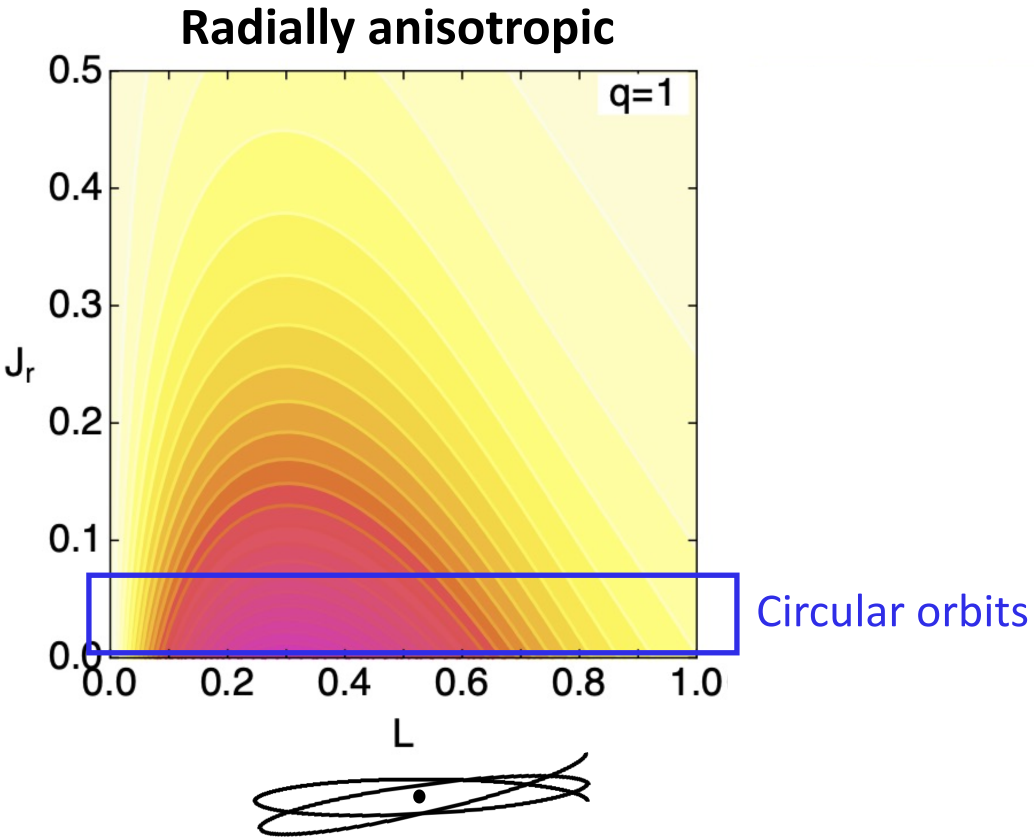

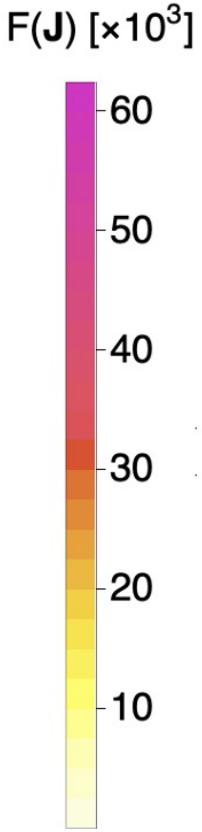

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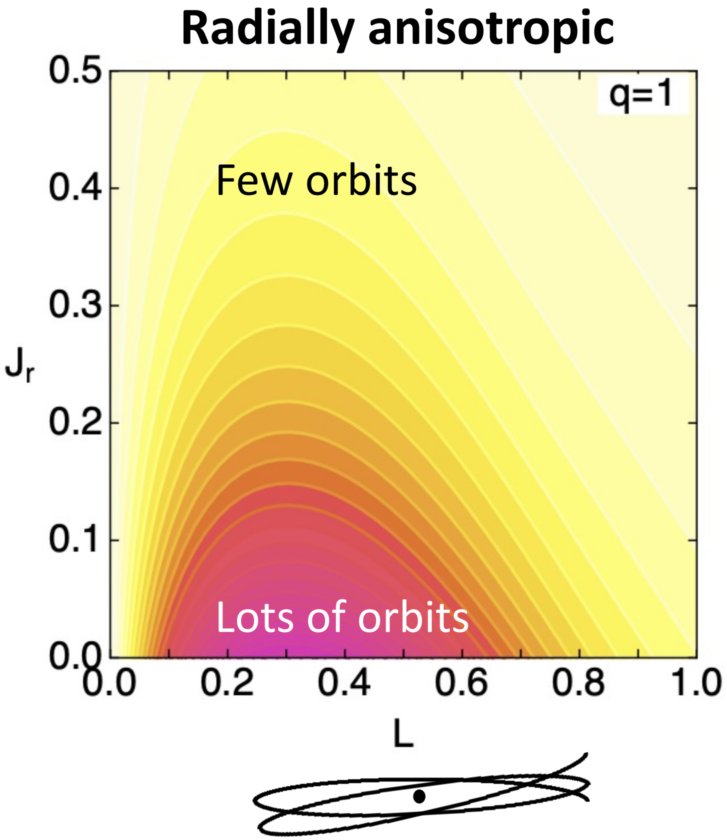

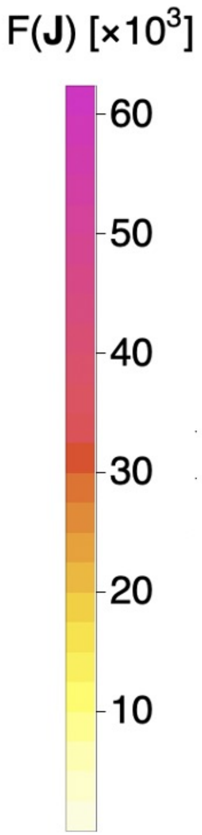

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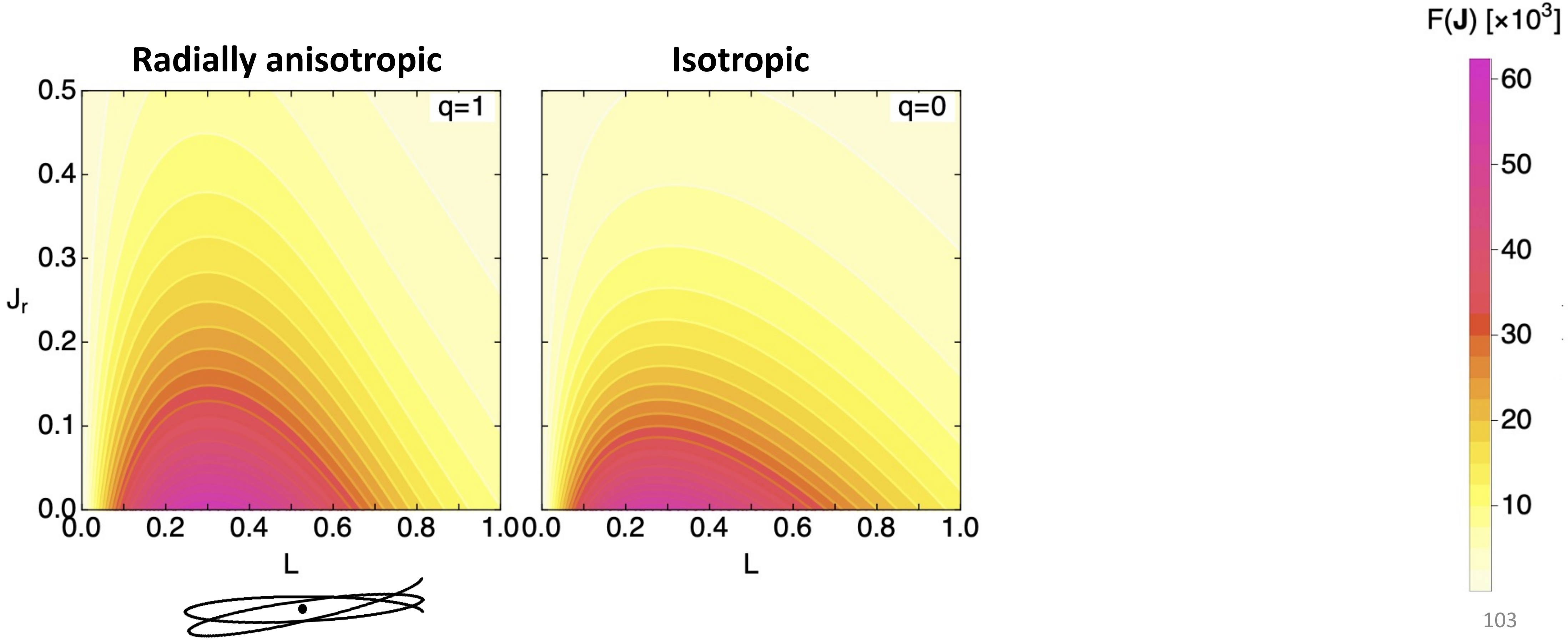

The Plummer cluster (distribution function)

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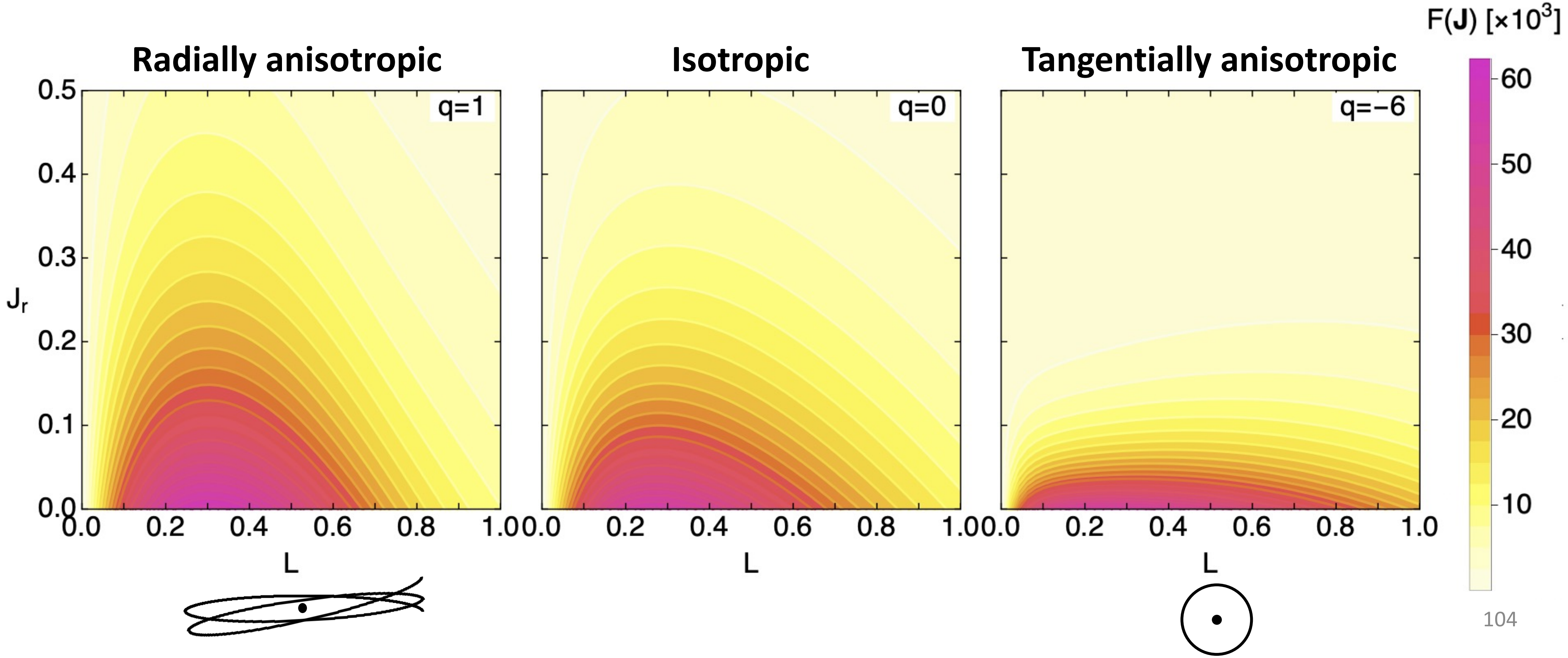
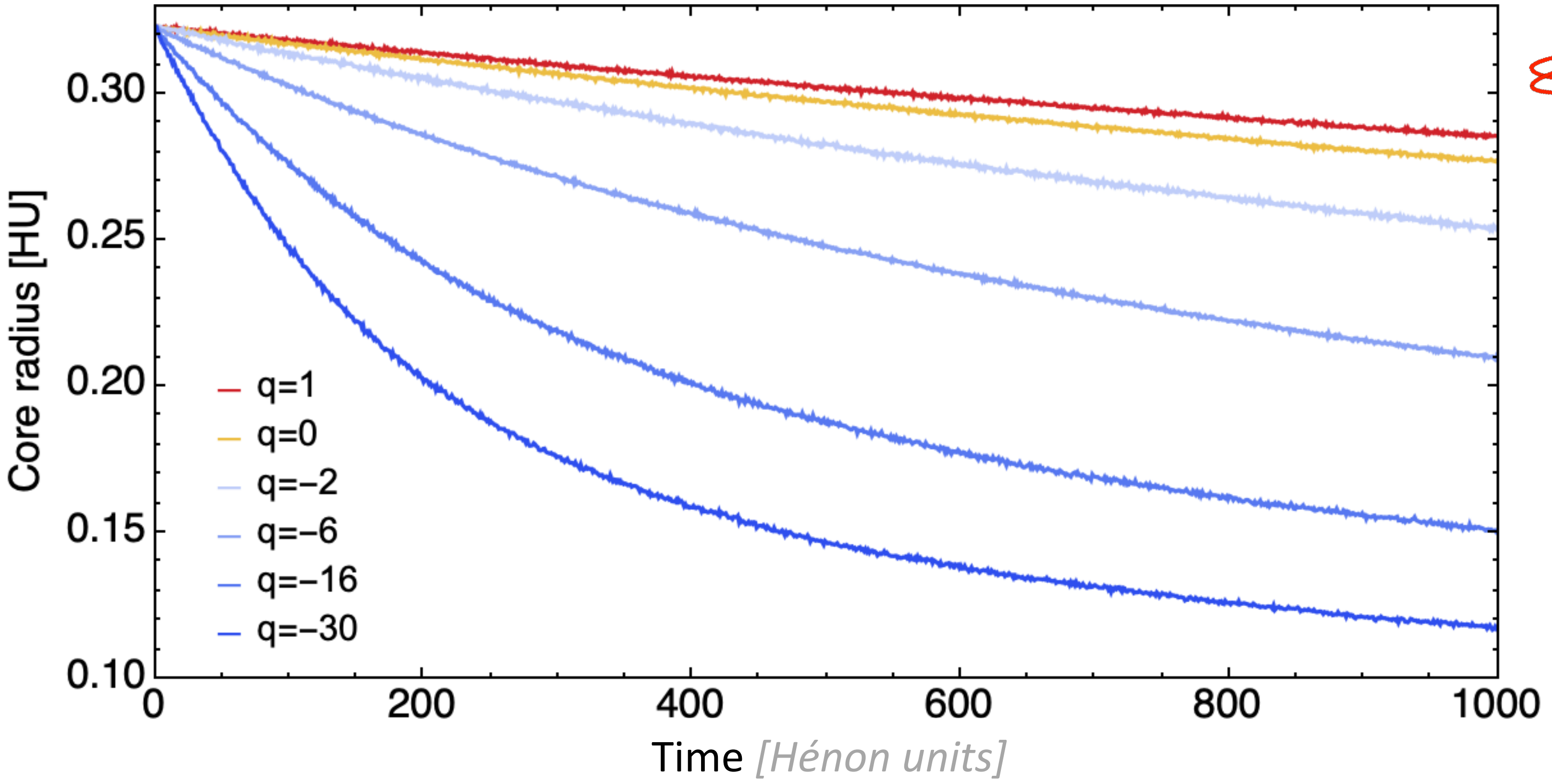


Core collapse vs anisotropy

- Numerical simulation: average over 100 realisations



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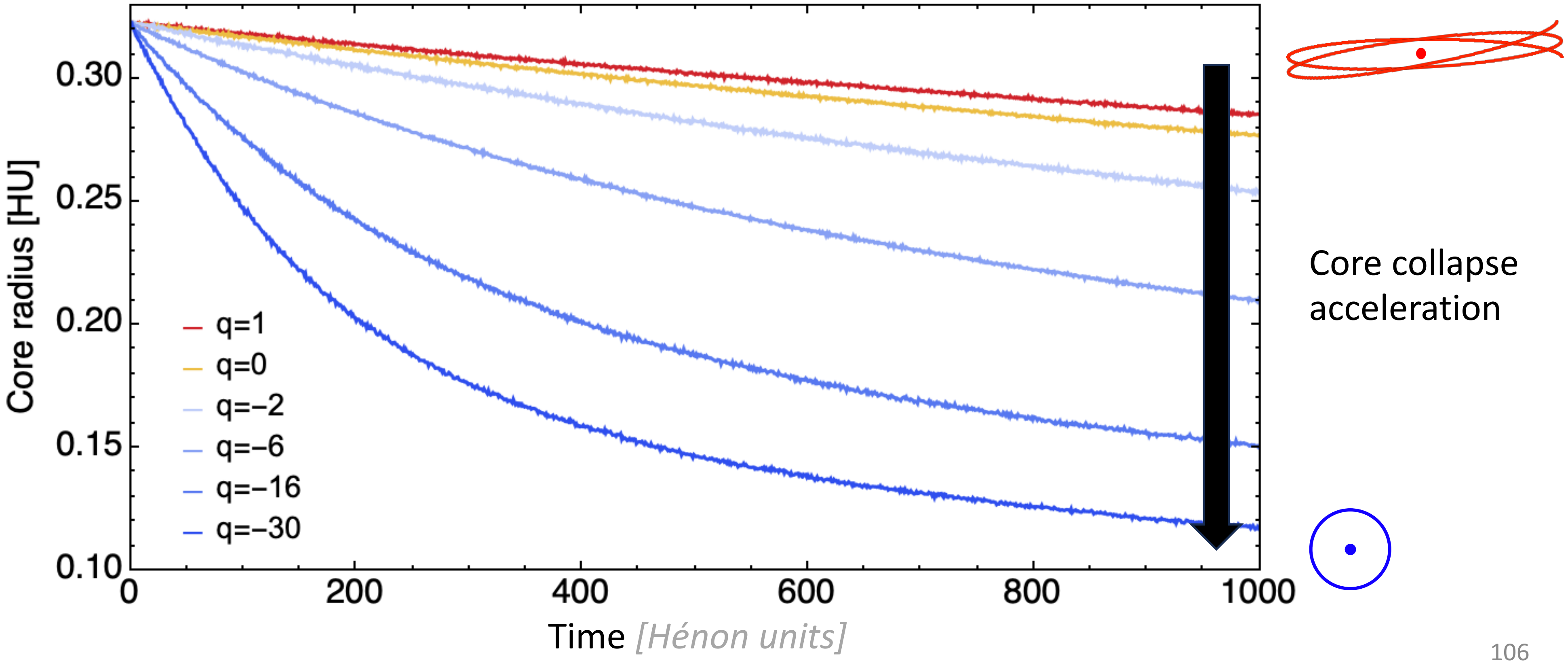
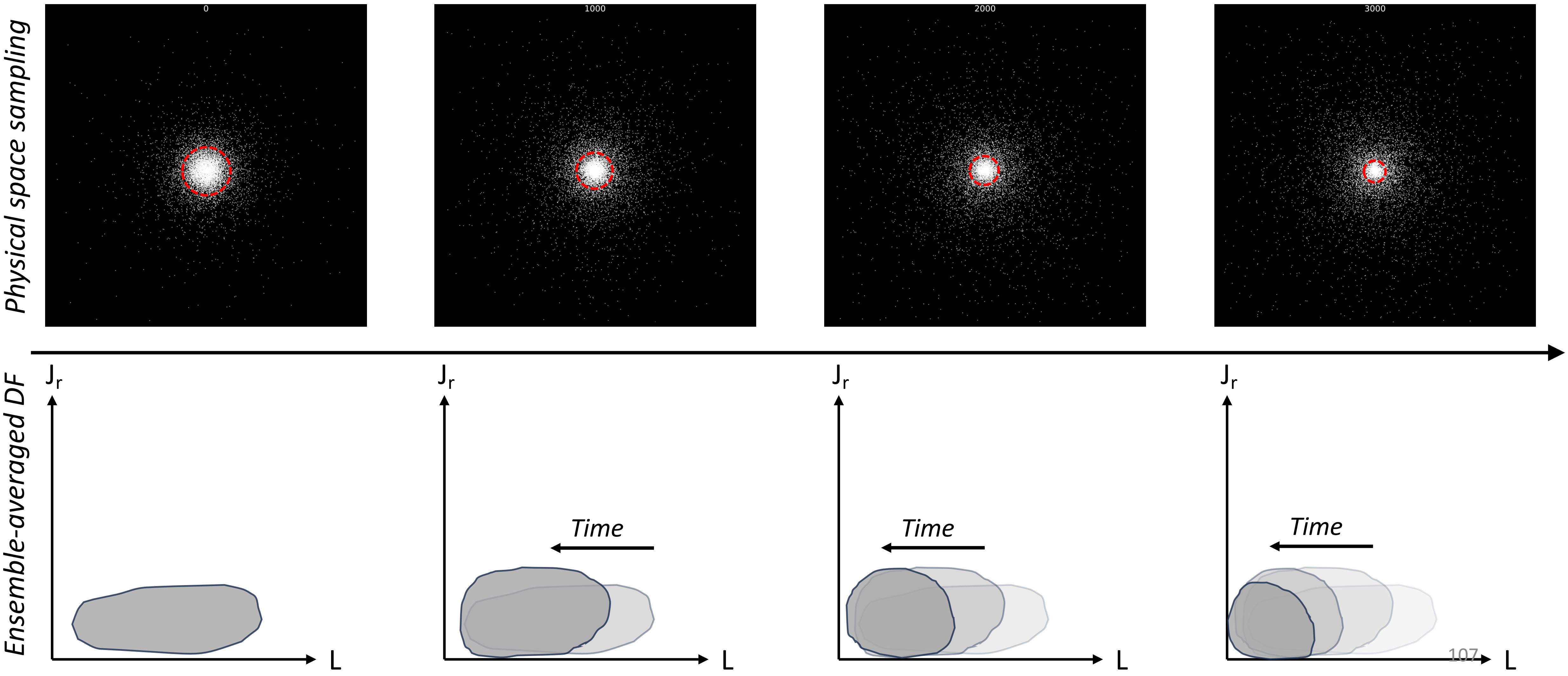
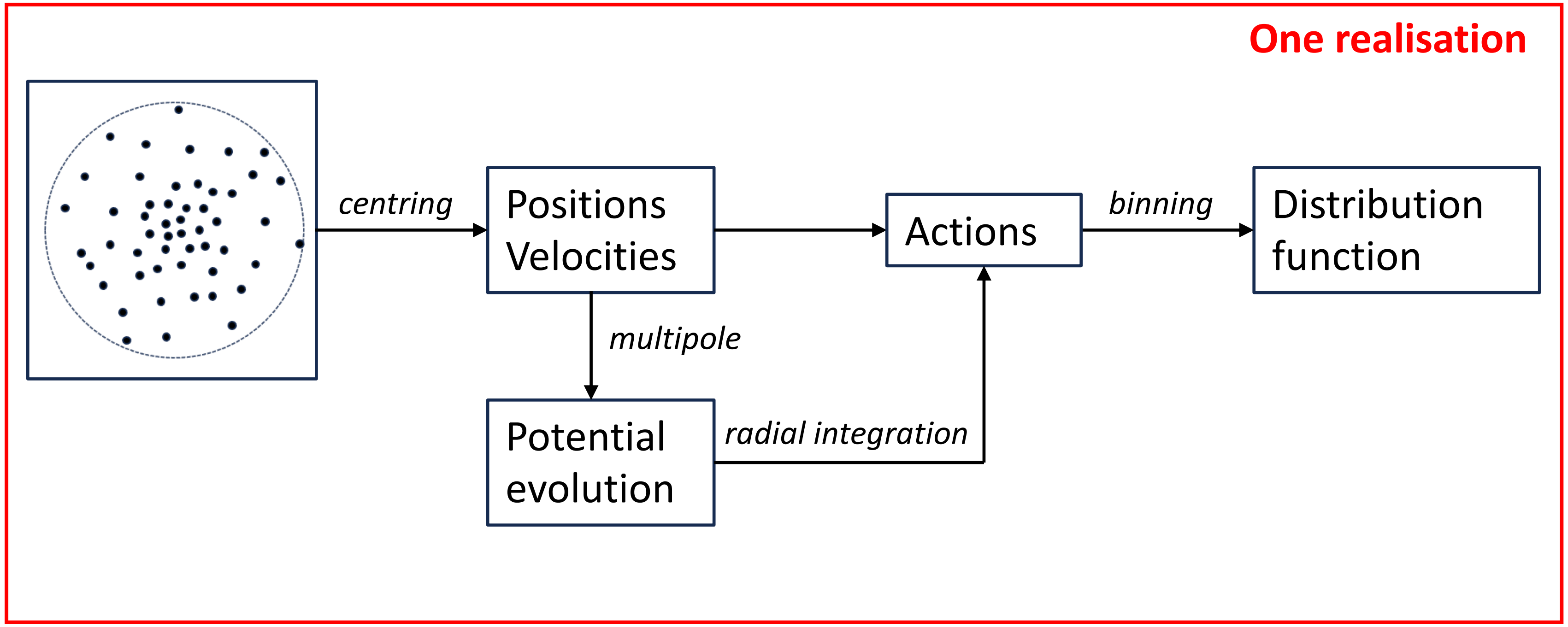


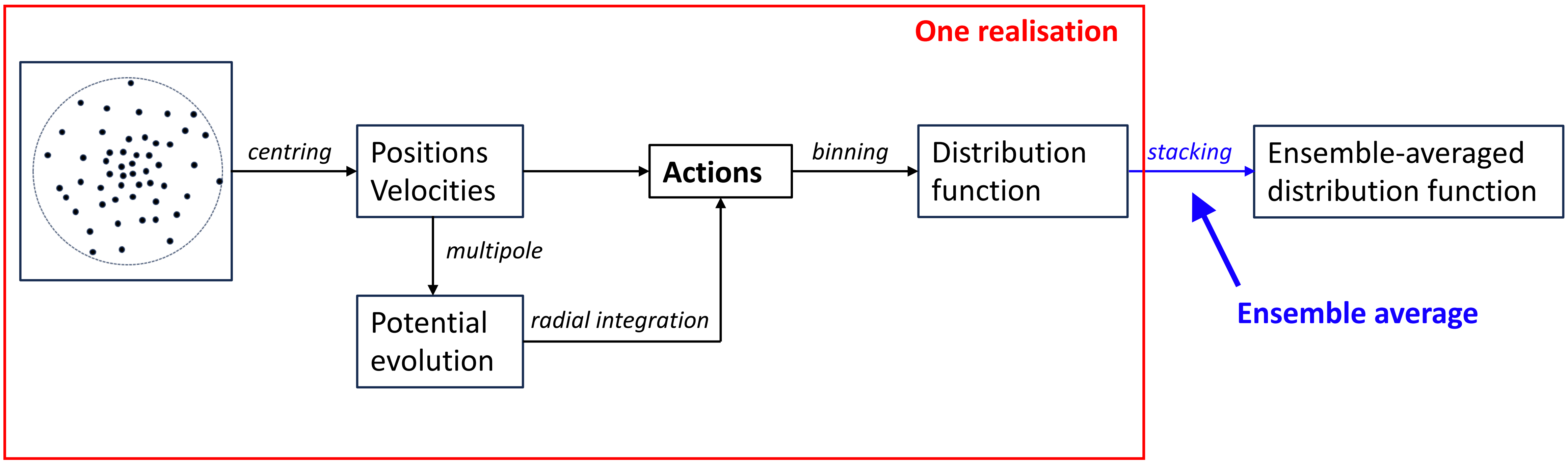
Configuration space vs orbital space



N-body prediction



N-body prediction



Theory for globular clusters

Heyvaerts (2010)

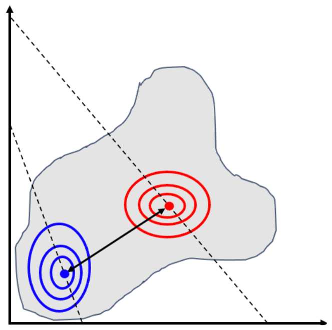
Balescu-Lenard (BL)

$$\sum_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J}) = \boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}')$$

$$\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}$$

$$\int \mathrm{d}\boldsymbol{J}'$$



No self-gravity →

Polyachenko & Shukhman (1982)
Chavanis (2012)

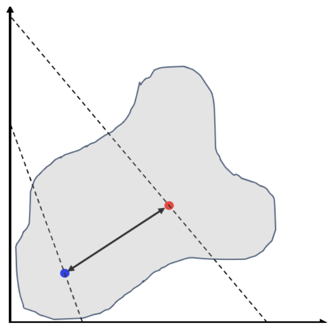
Landau (RR)

$$\sum_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J}) = \boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}')$$

$$\psi_{\boldsymbol{k}\boldsymbol{k}'}$$

$$\int \mathrm{d}\boldsymbol{J}'$$



Local homogeneity →

Chandrasekhar (1943)
Chavanis (2013)

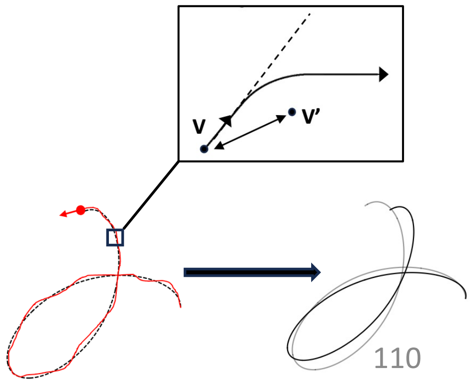
Orbit-averaged Chandrasekhar

$$\int \mathrm{d}\boldsymbol{k}$$

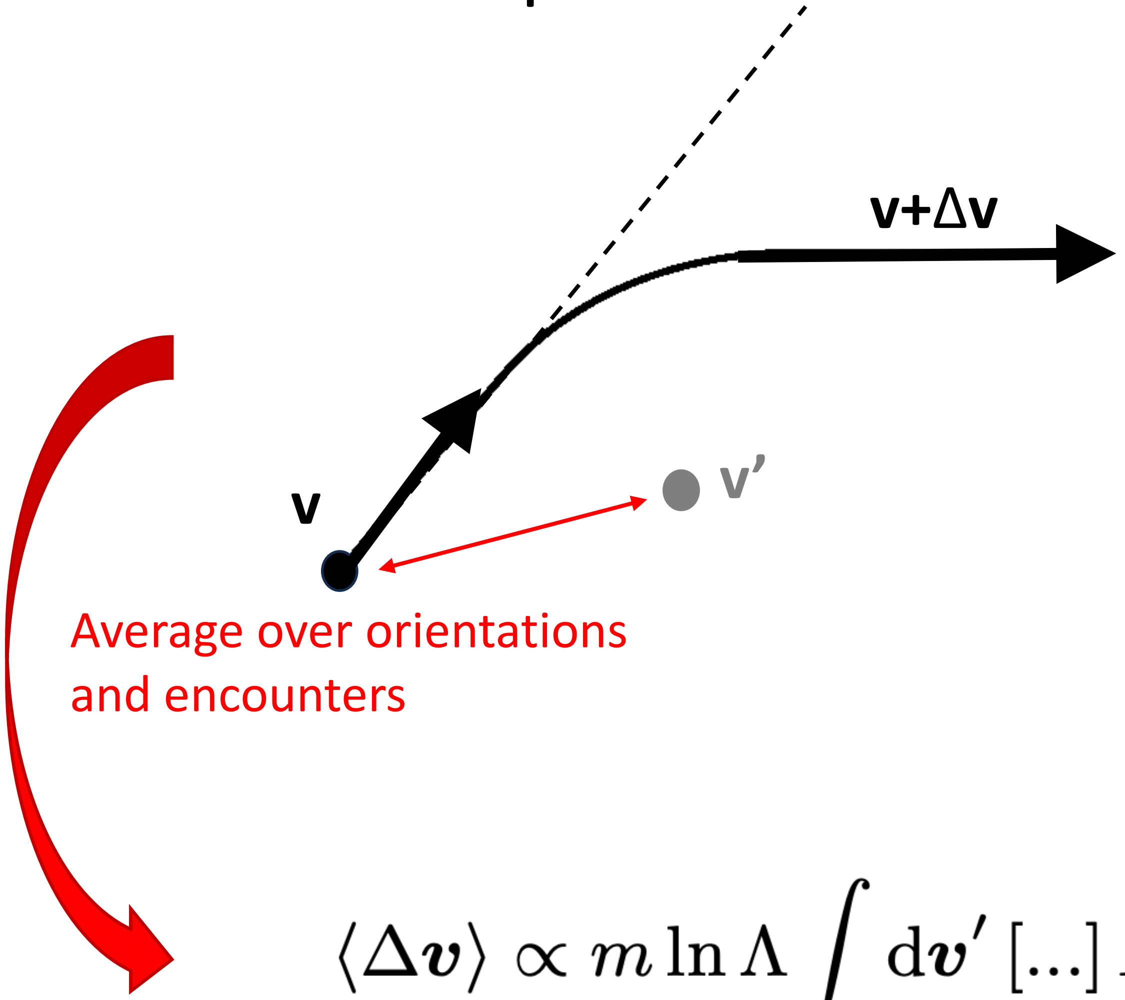
$$\boldsymbol{k}\cdot\boldsymbol{v} = \boldsymbol{k}\cdot\boldsymbol{v}'$$

$$\widehat{u}(\mathbf{k})$$

$$\int \mathrm{d}\boldsymbol{v}'$$

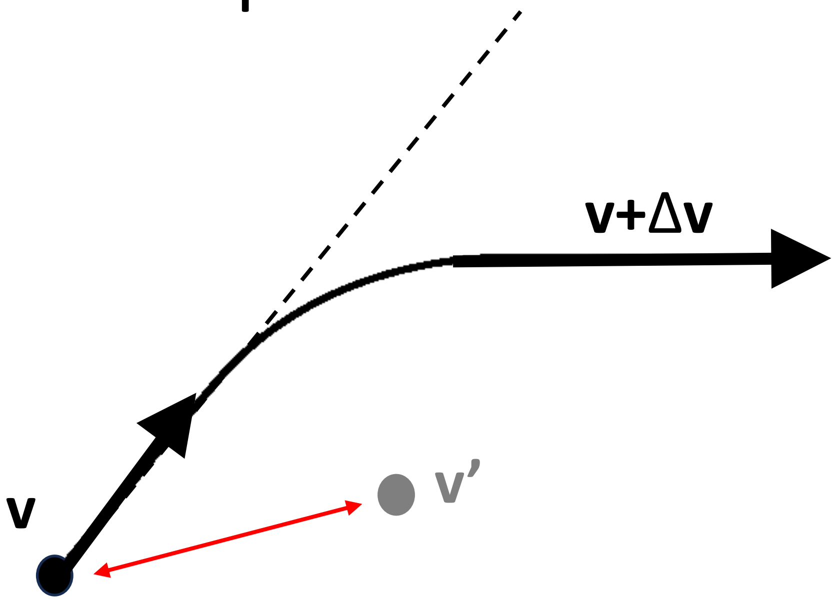


Theoretical prediction: Chandrasekhar theory



$$\langle \Delta \boldsymbol{v} \rangle \propto m \ln \Lambda \int \mathrm{d}\boldsymbol{v}' \, [\ldots] \, F(\boldsymbol{v}')$$

Theoretical prediction: Chandrasekhar theory

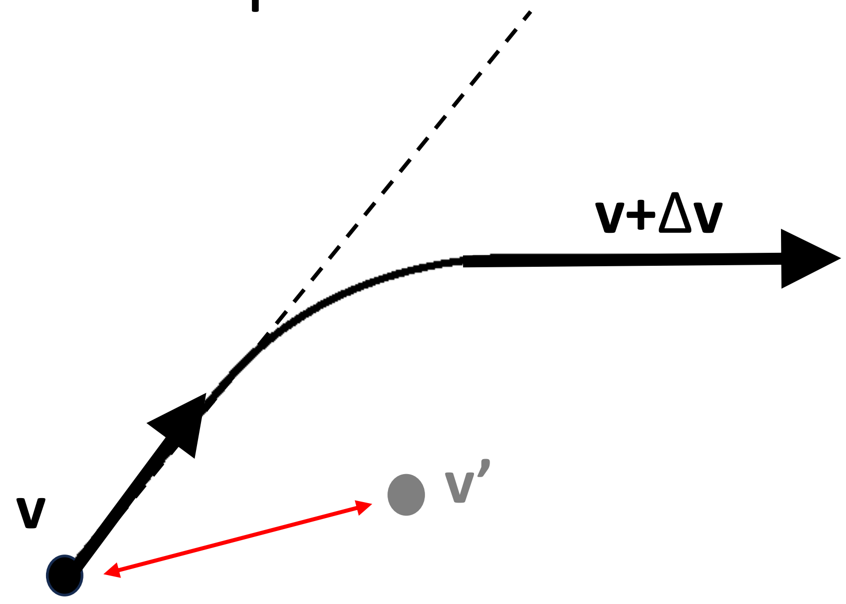


Average over orientations and encounters

$$\langle \Delta \boldsymbol{v} \rangle \propto m \ln \Lambda \int \mathrm{d}\boldsymbol{v}' \, [...] \, F(\boldsymbol{v}')$$

Fluctuations 1/N

Theoretical prediction: Chandrasekhar theory



Average over orientations and encounters

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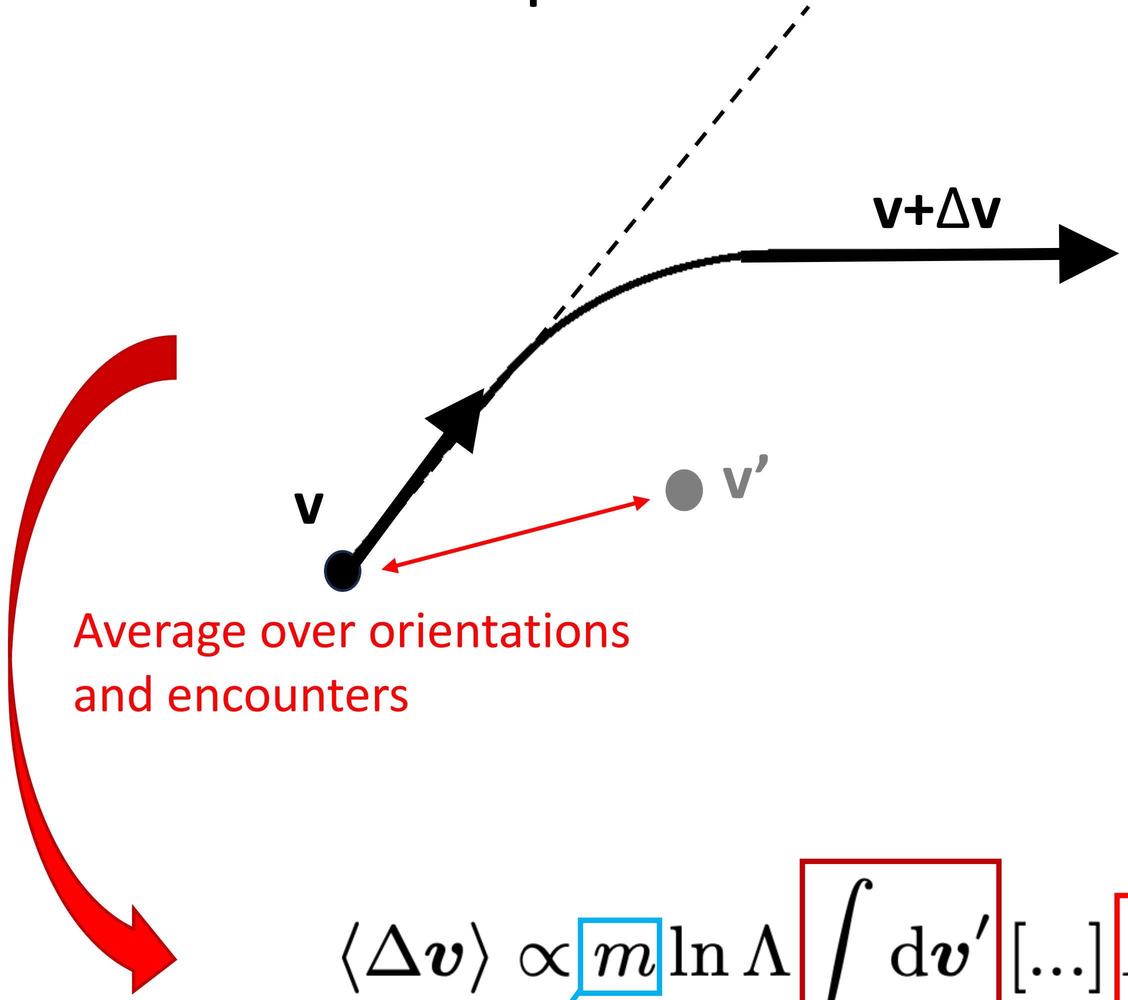
Fluctuations 1/N

DF of perturbers

Theoretical prediction: Chandrasekhar theory



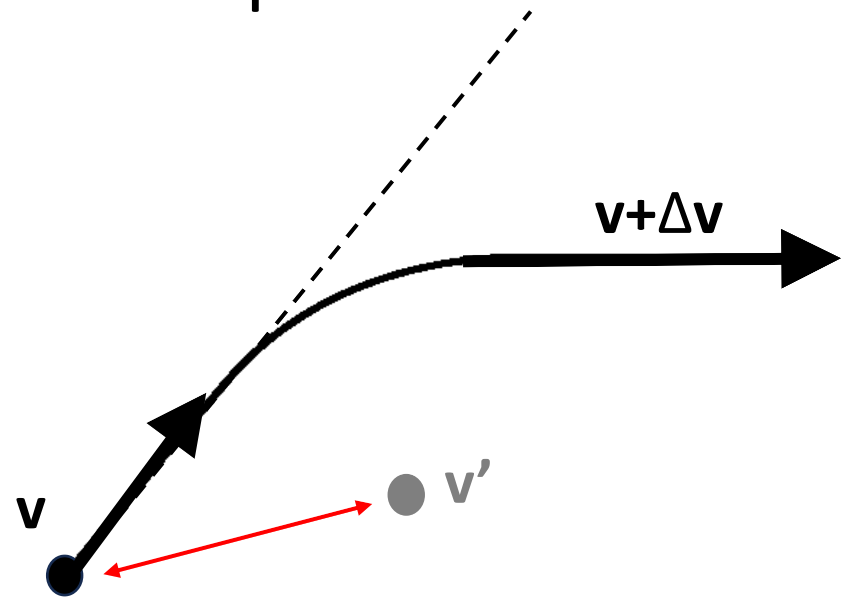
$$\langle \Delta \boldsymbol{v} \rangle \propto m \ln \Lambda \int \mathrm{d}\boldsymbol{v}' [\ldots] F(\boldsymbol{v}')$$

Fluctuations 1/N

Scan over perturbers

DF of perturbers

Theoretical prediction: Chandrasekhar theory

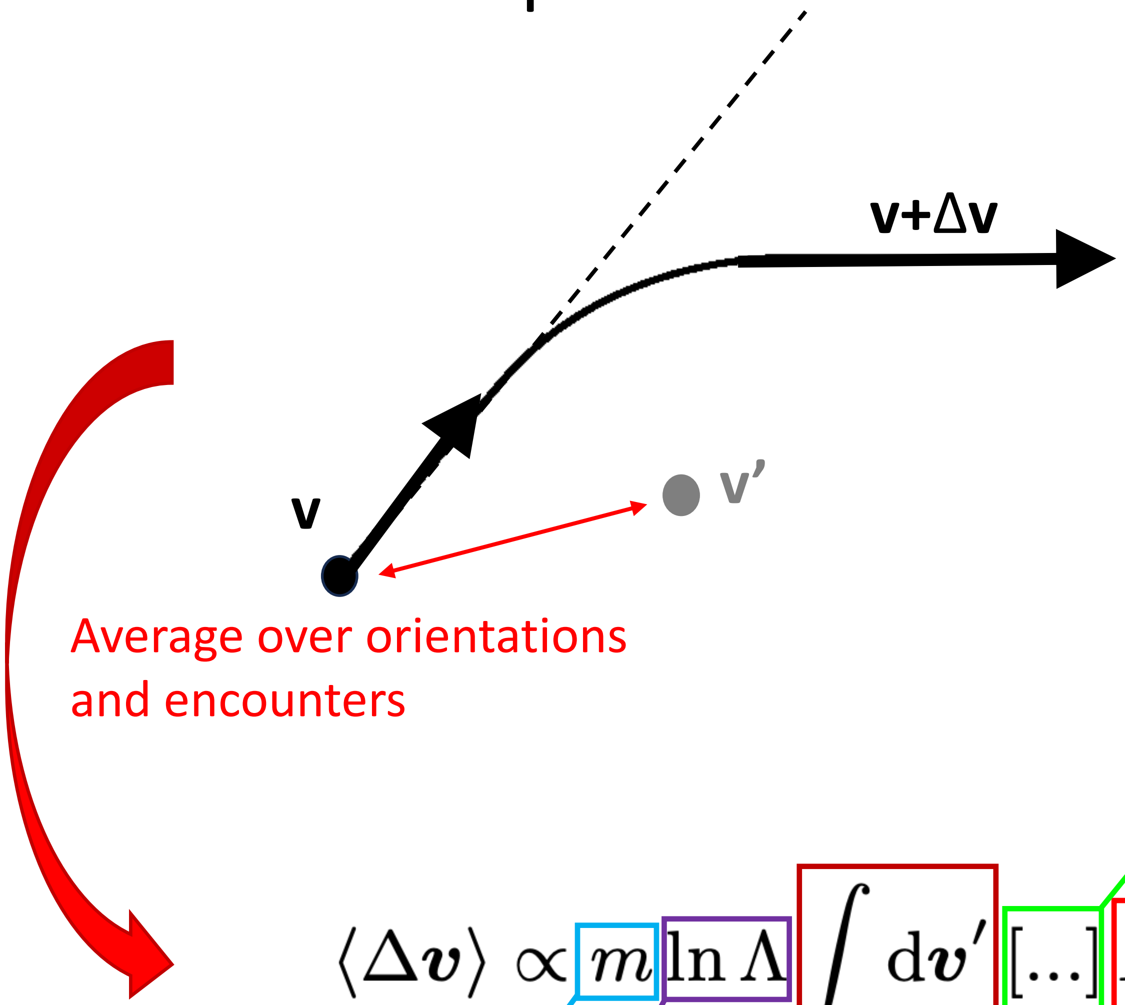


Average over orientations and encounters

$$\langle \Delta \boldsymbol{v} \rangle \propto m \ln \Lambda \int \mathrm{d}\boldsymbol{v}' \, [\ldots] \, F(\boldsymbol{v}')$$

- m: Fluctuations 1/N
- $\ln \Lambda$: Coulomb logarithm
- $\int \mathrm{d}\boldsymbol{v}'$: Scan over perturbers
- $F(\boldsymbol{v}')$: DF of perturbers

Theoretical prediction: Chandrasekhar theory



$$\langle \Delta \boldsymbol{v} \rangle \propto m \ln \Lambda \int \mathrm{d}\boldsymbol{v}' \, [\ldots] \, F(\boldsymbol{v}')$$

Generalise to anisotropic clusters *(Tep et al. 2022)*

Fluctuations 1/N

Coulomb logarithm

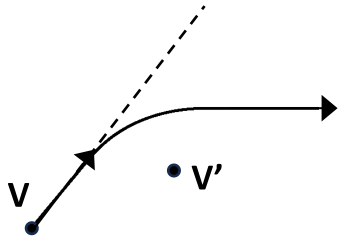
Scan over perturbers

DF of perturbers

Theoretical prediction: Chandrasekhar theory

Velocity deflections

3 integrations

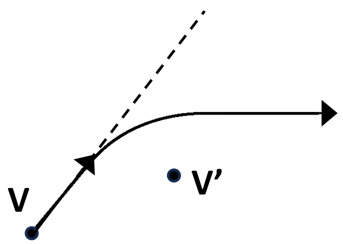


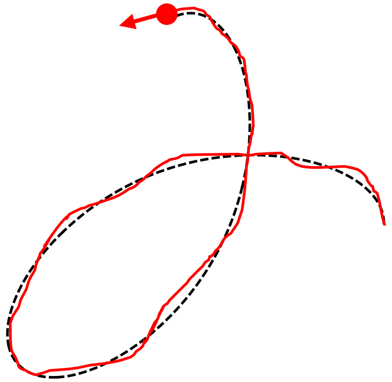
Theoretical prediction: Chandrasekhar theory

Velocity deflections → 2nd order variation → Motion integral diffusion

3 integrations

2D inversion





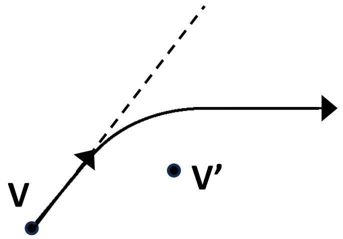
Theoretical prediction: Chandrasekhar theory

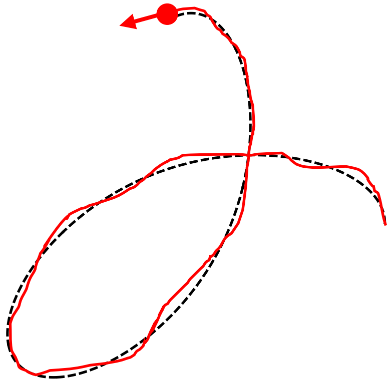
Velocity deflections → (2^{nd} order variation) → Motion integral diffusion → (Orbit-average) → Orbit-averaged diffusion coefficients

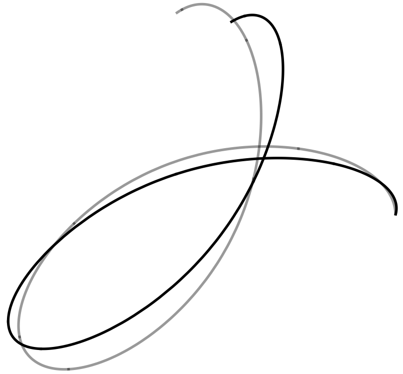
3 integrations

2D inversion

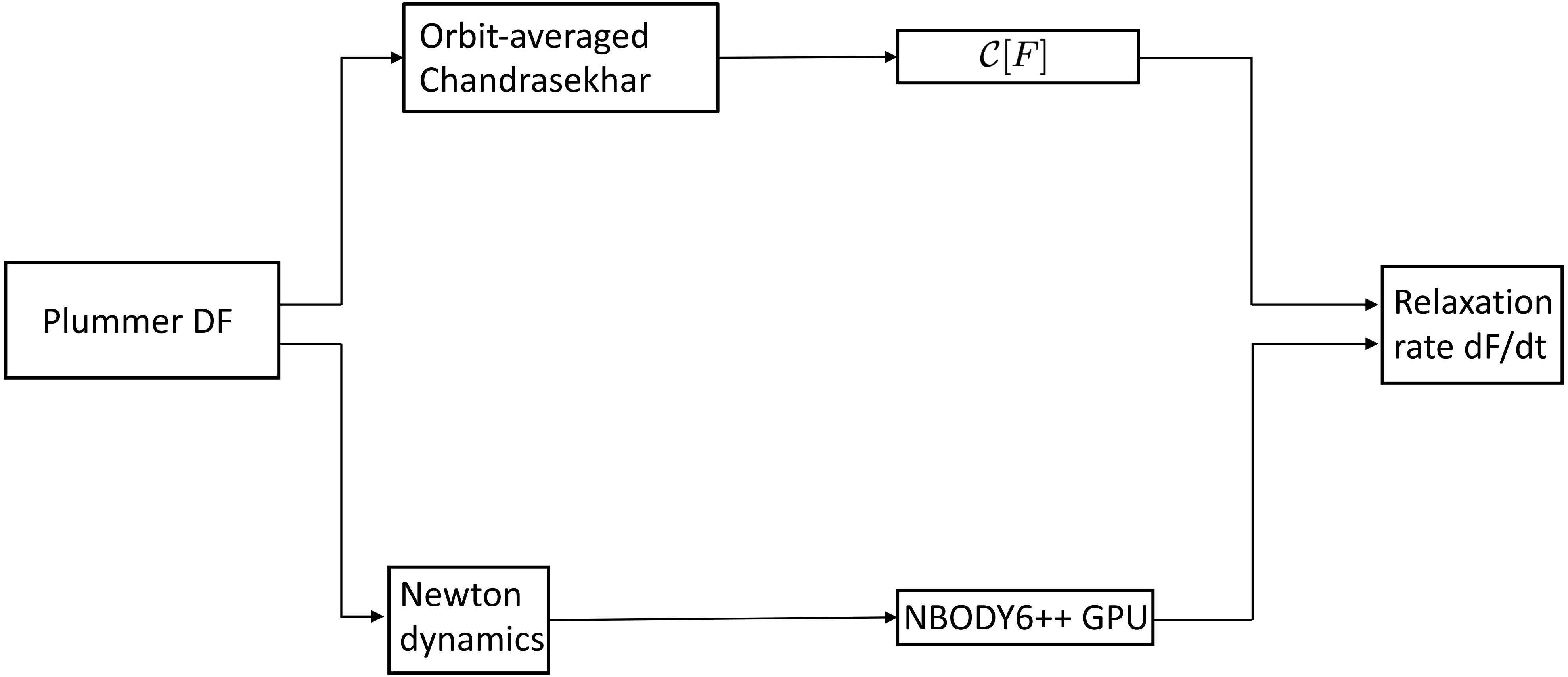
1 integration
2D sampling



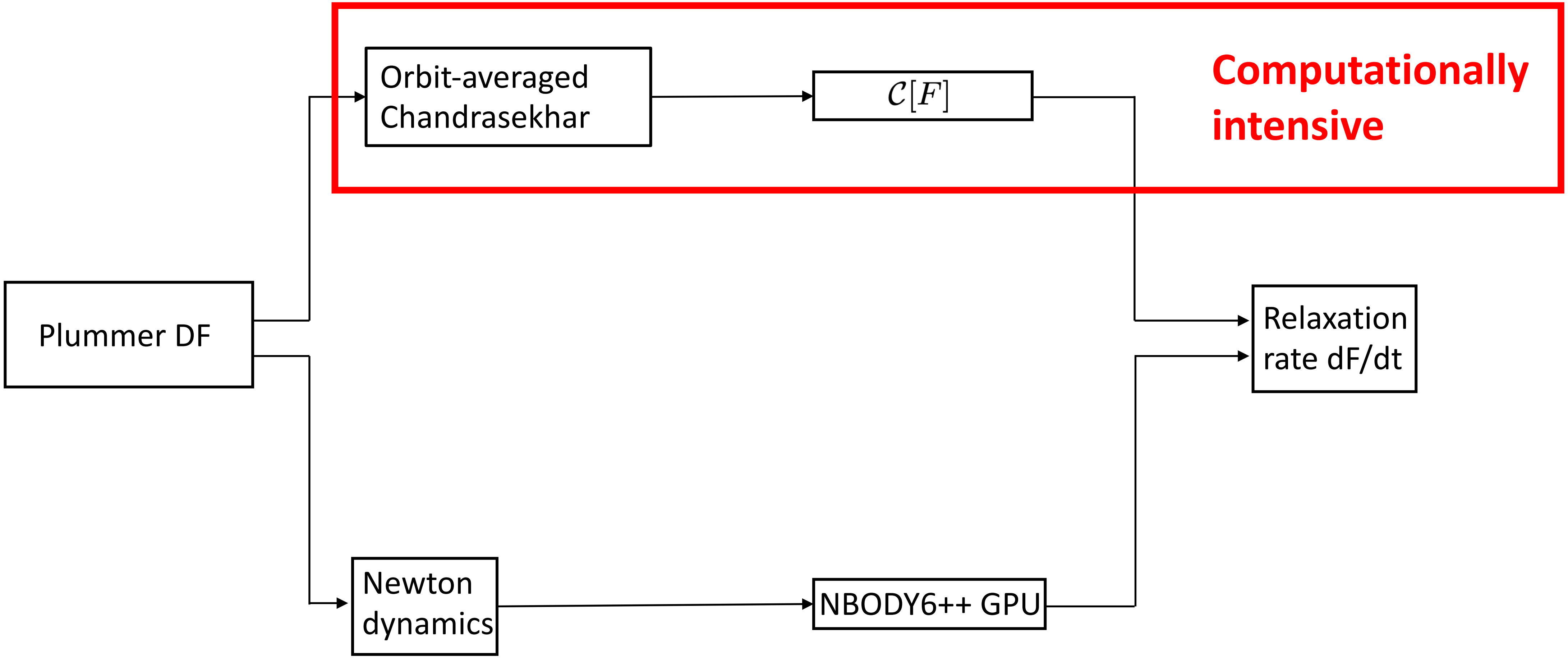




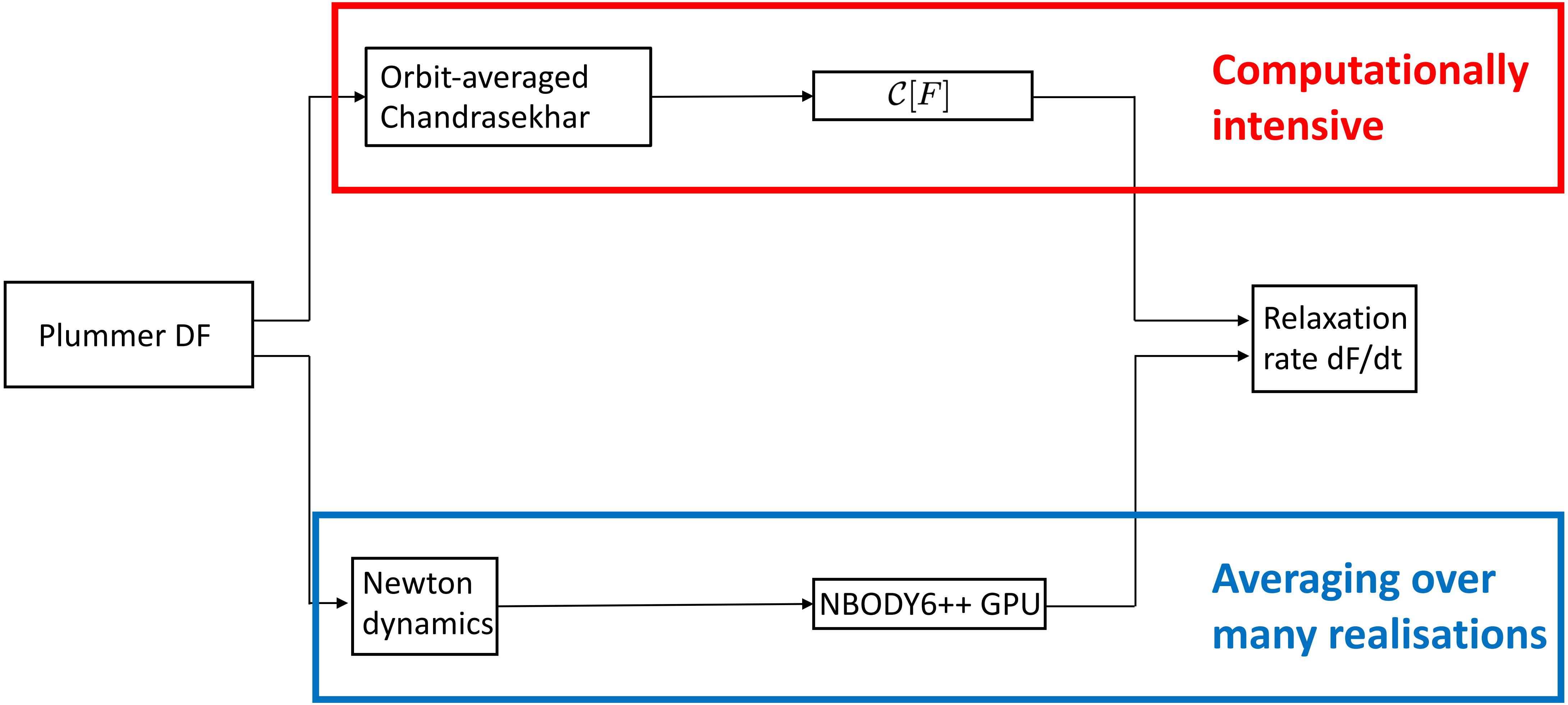
Secular response prediction



Secular response prediction

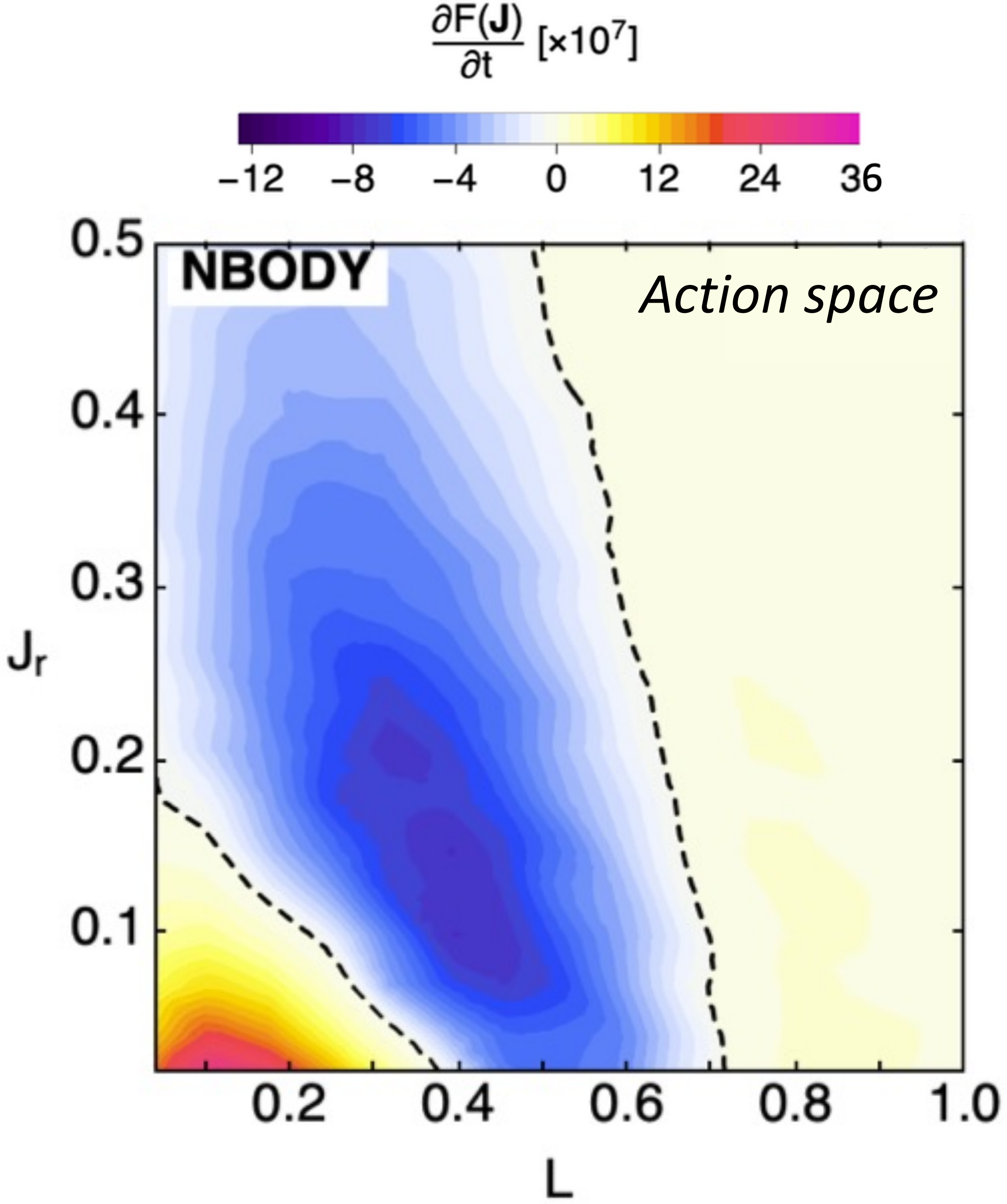


Secular response prediction



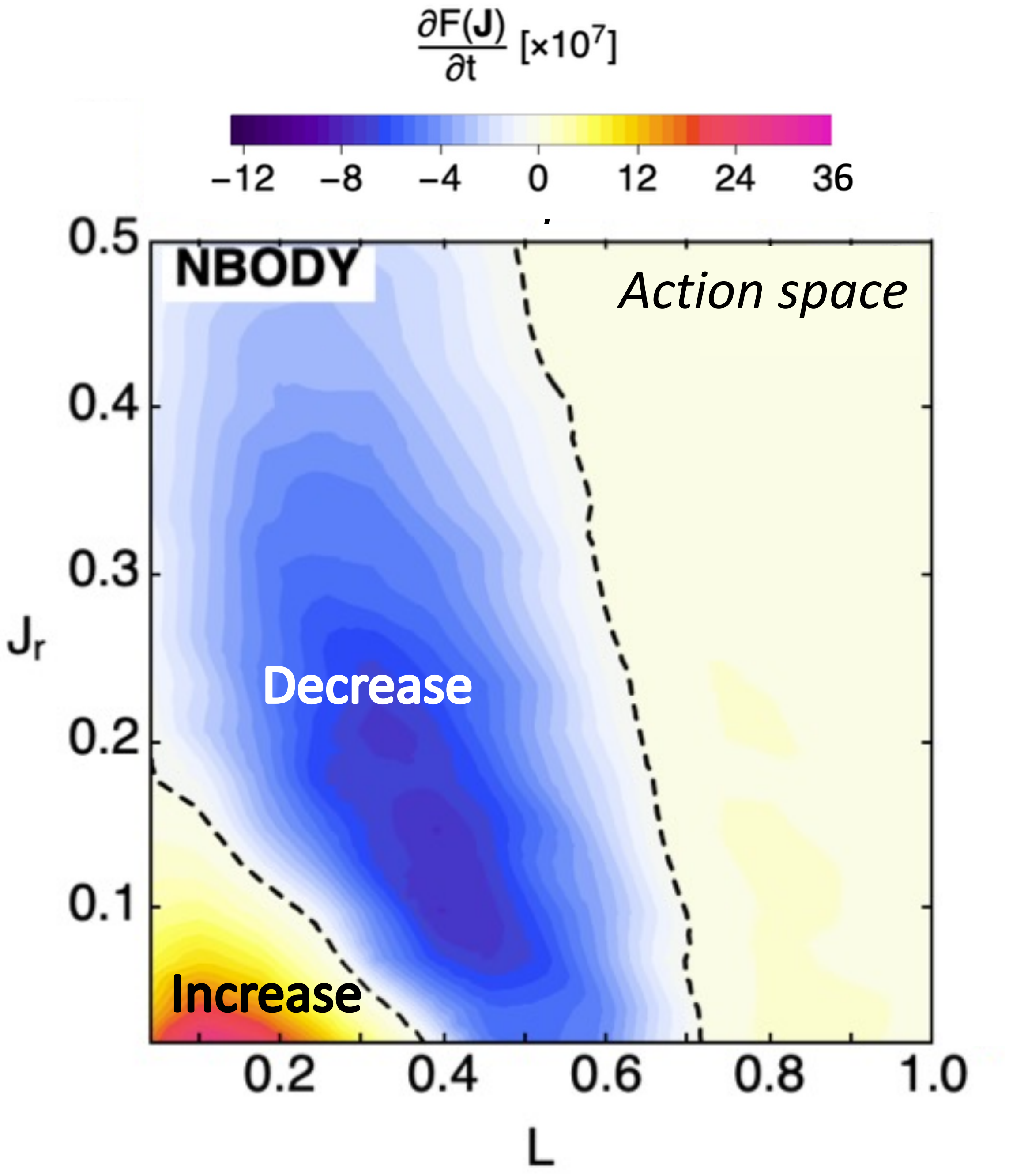
Orbital diffusion

- Relaxation rate: $dF/dt_{|t=0^+}$



Orbital diffusion

- Relaxation rate: $dF/dt_{|t=0^+}$

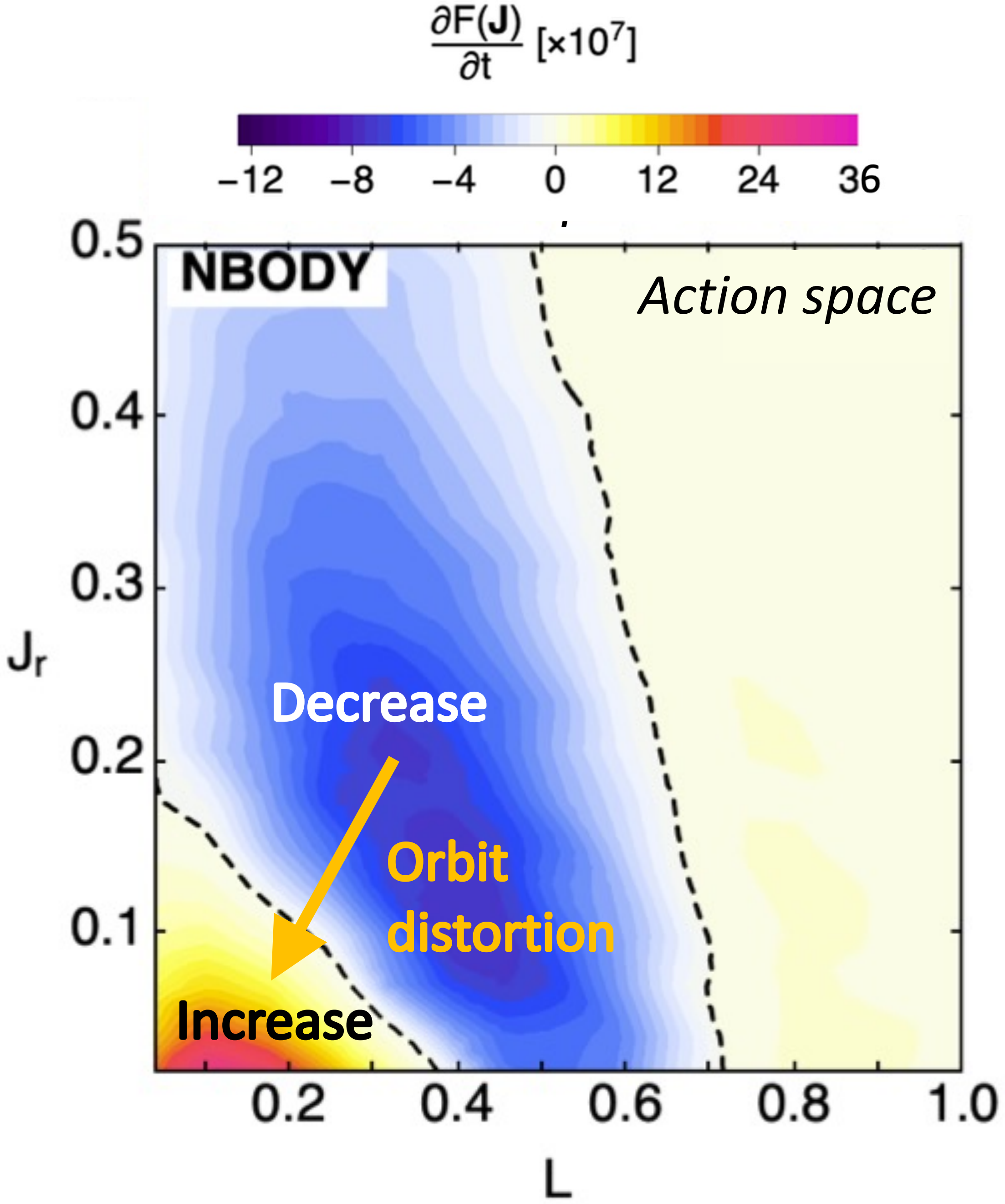


100 realisations

Orbital diffusion

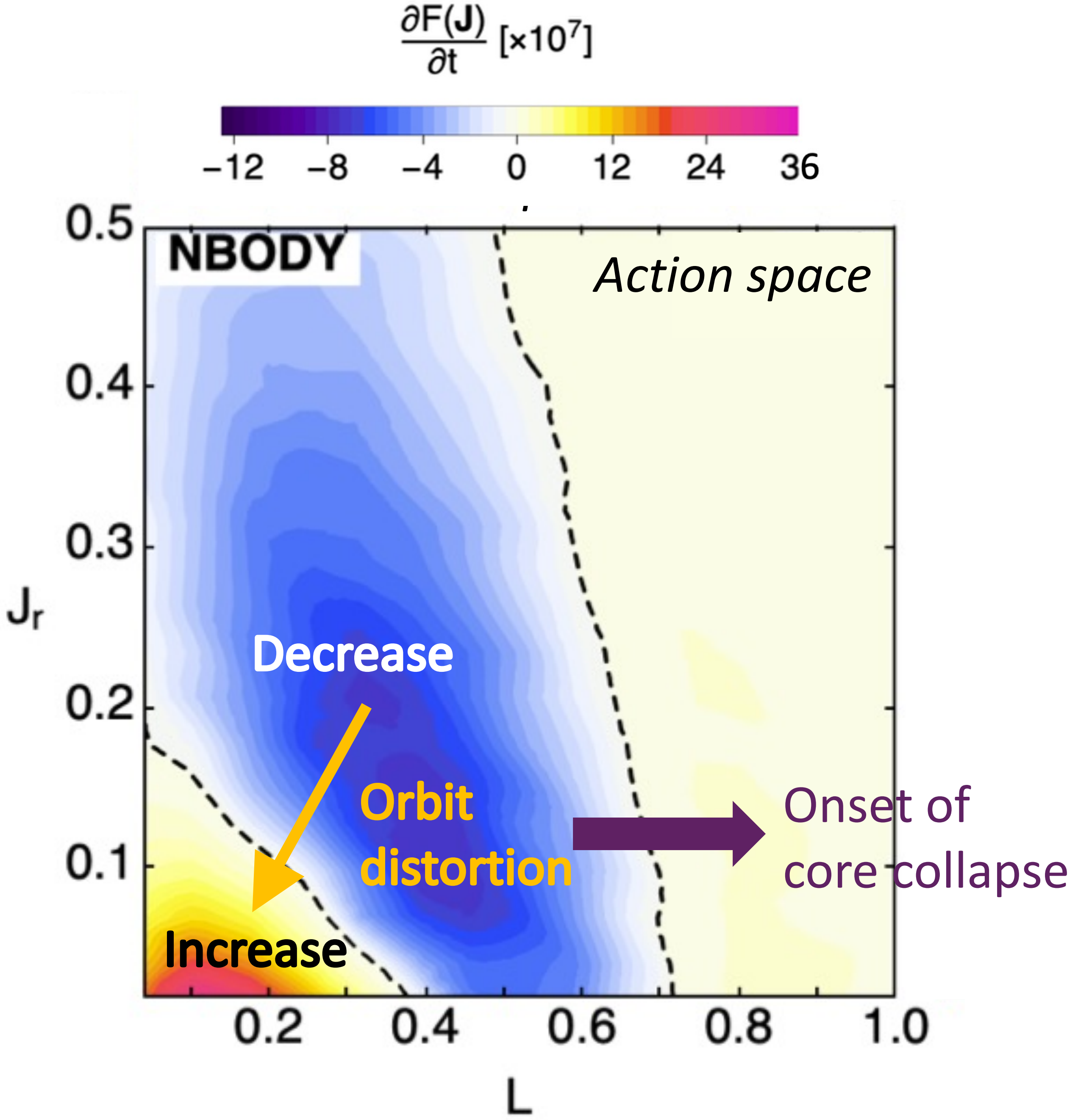
- Relaxation rate: $dF/dt_{|t=0^+}$

100 realisations



Orbital diffusion

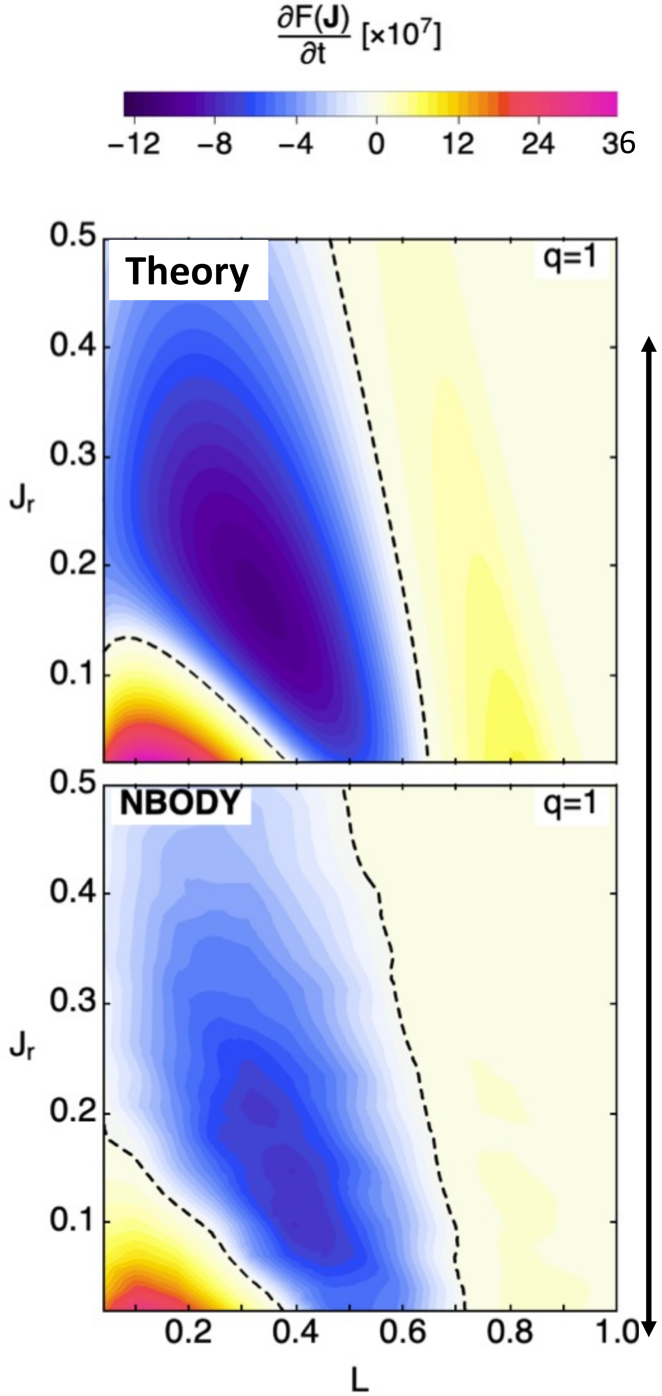
- Relaxation rate: $dF/dt_{|t=0^+}$



100 realisations

Orbital diffusion

- Theoretical prediction
- N-body measurement

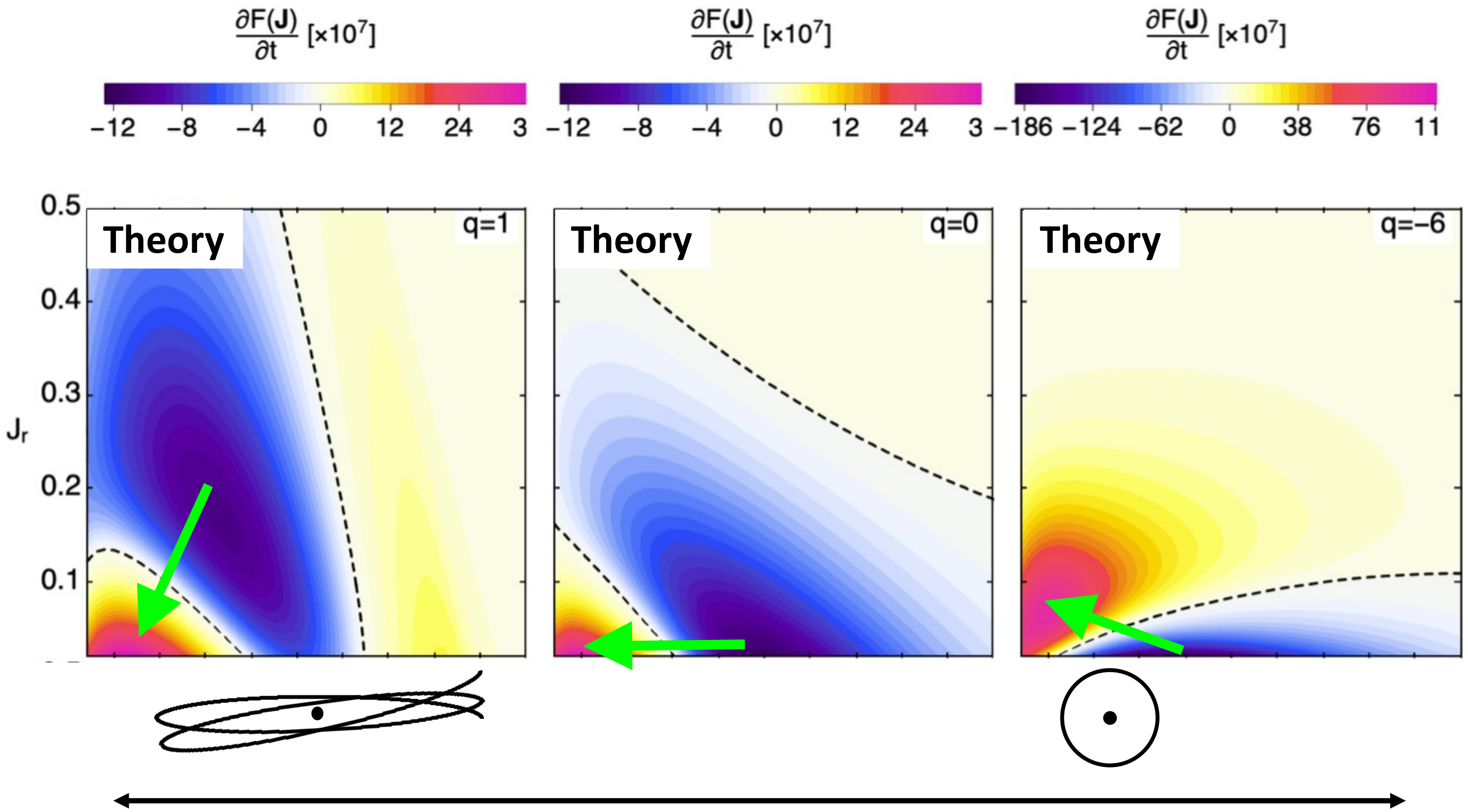


Qualitative agreement between Theory and NBODY simulations

Up to overall prefactor
(Darker colors for theory)

Orbital diffusion

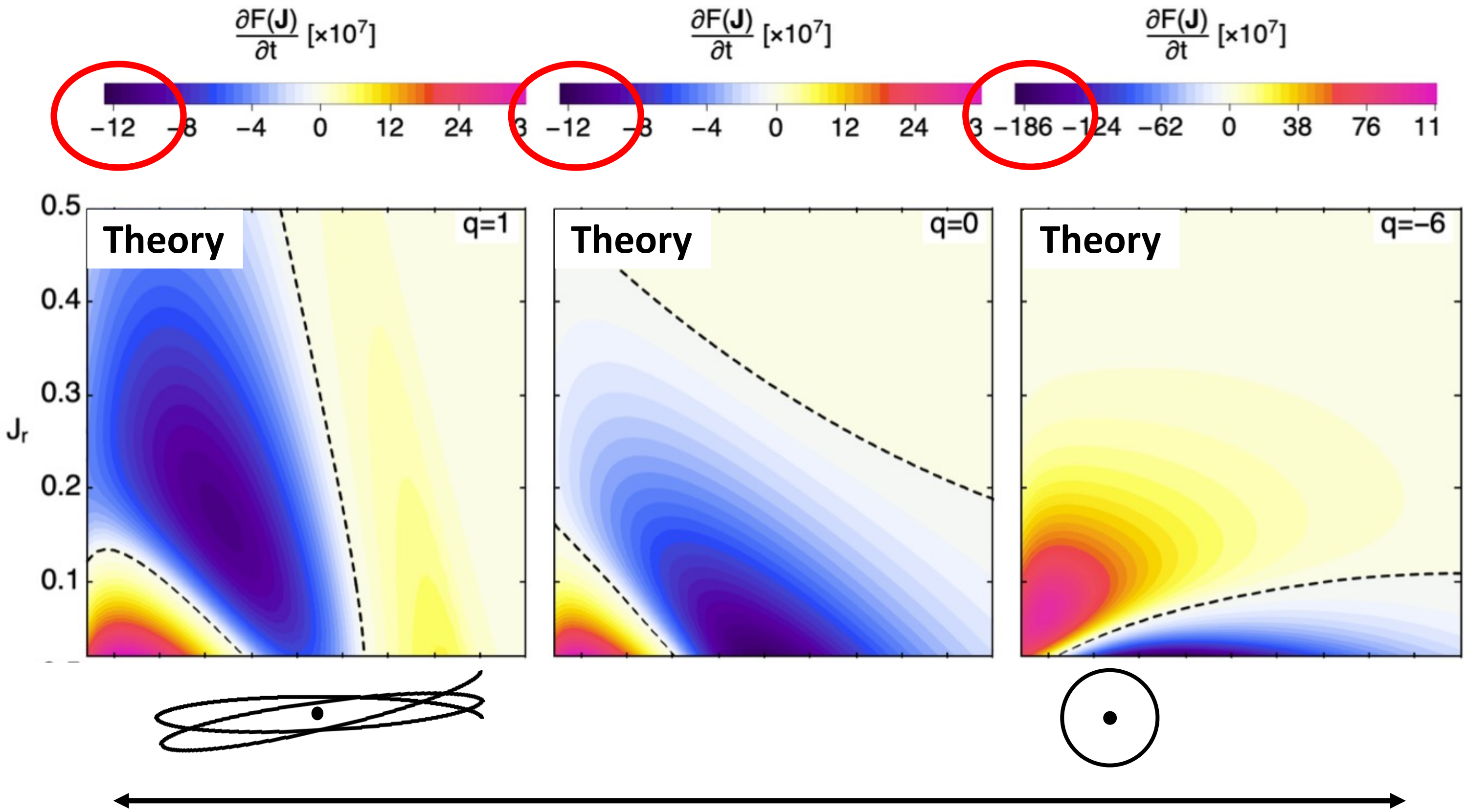
- Isotropisation vs anisotropy
- Orbital reshuffling



Impact of anisotropy on the rate of orbital change

Orbital diffusion

- Core collapse acceleration
- Orbital reshuffling

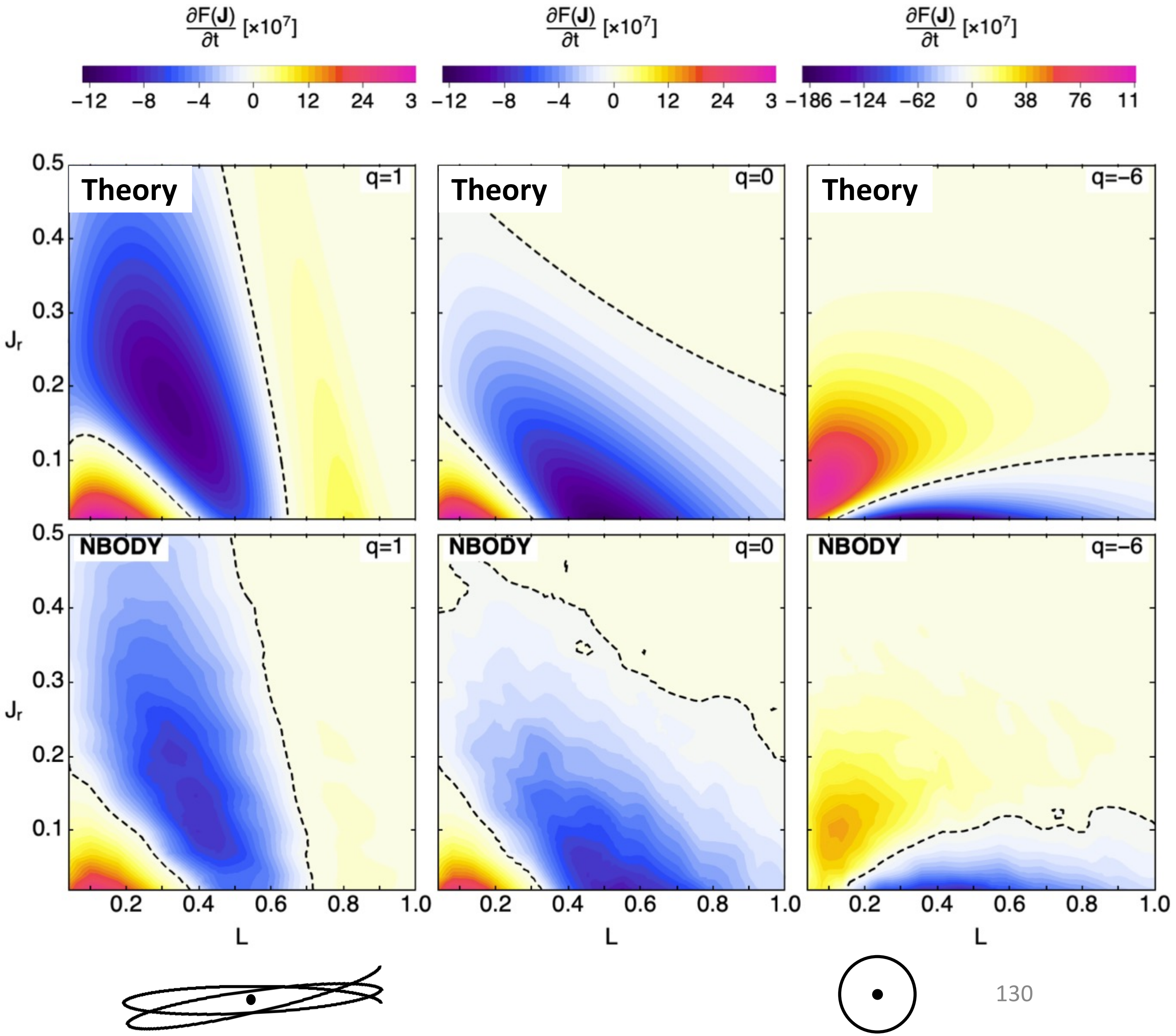


Impact of anisotropy on the rate of orbital change

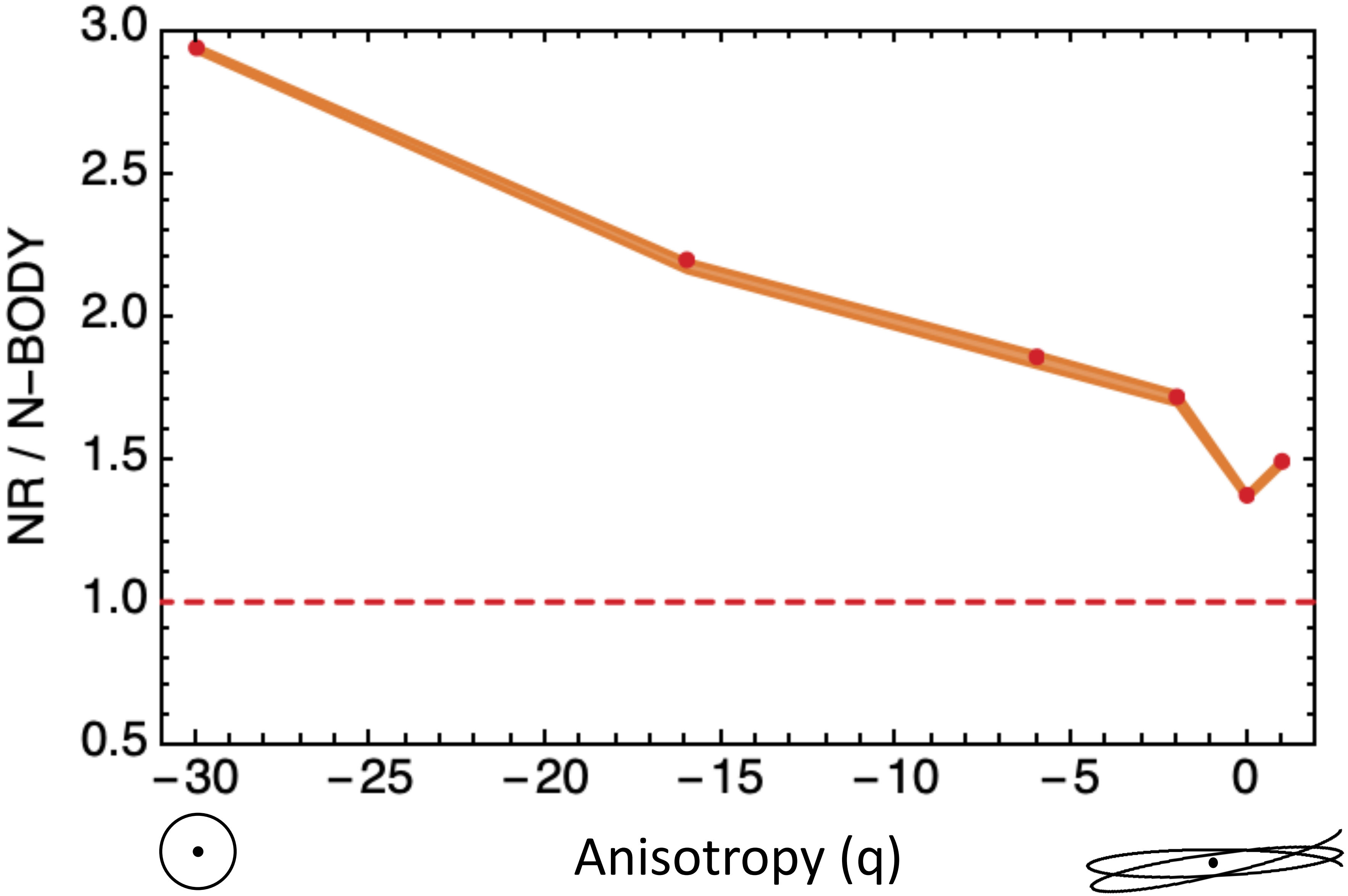
Orbital diffusion

- Isotropisation
- Core collapse acceleration

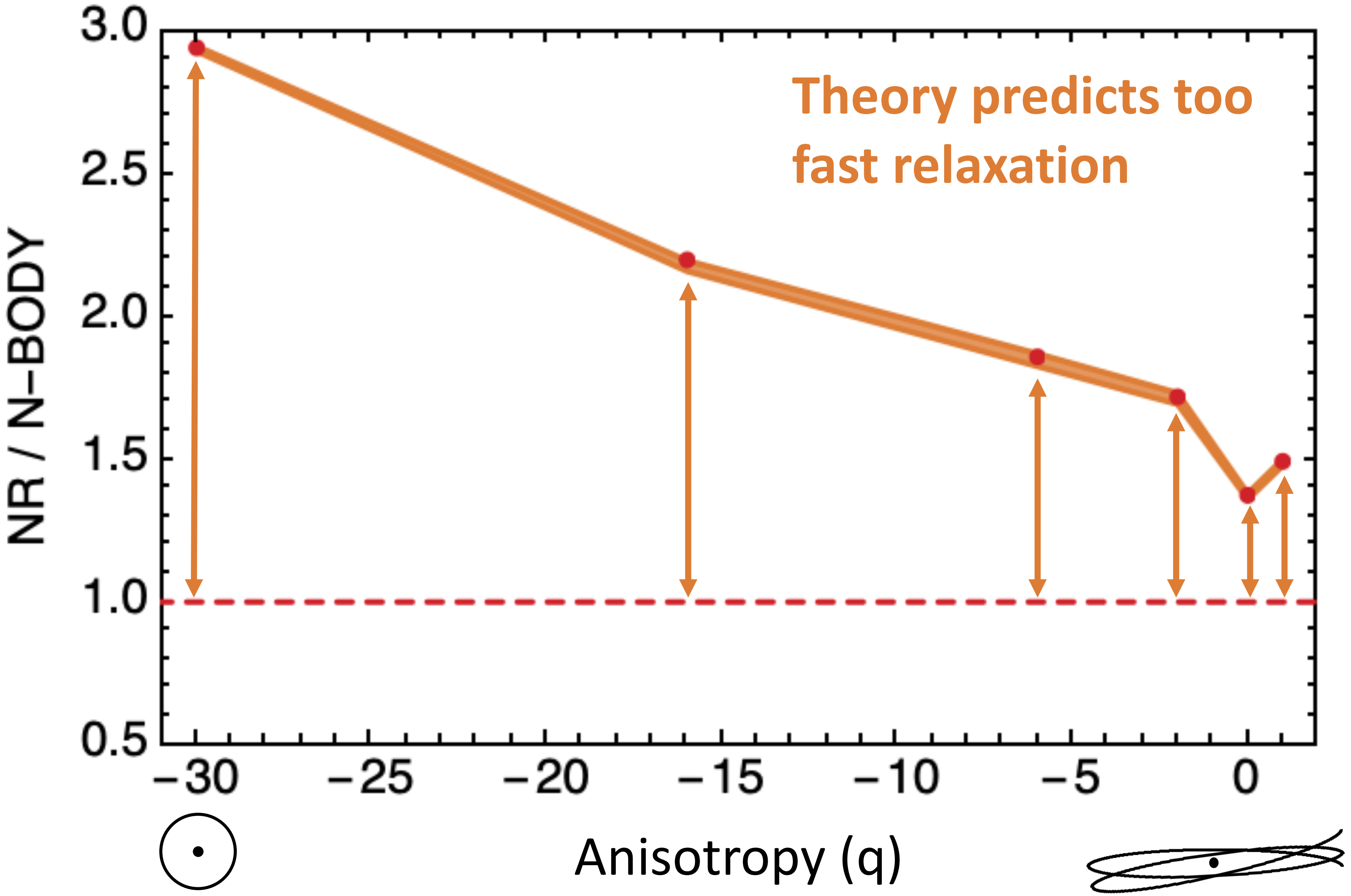
Satisfying prediction



Limits of the Chandrasekhar approach



Limits of the Chandrasekhar approach



What about global resonances?

Heyvaerts (2010)

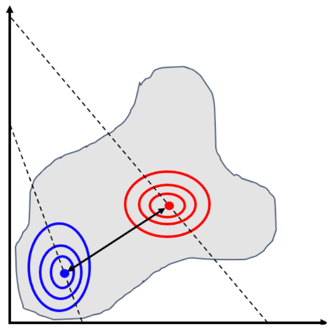
Balescu-Lenard (BL)

$$\sum_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})=\boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}')$$

$$\psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}$$

$$\int \mathrm{d}\boldsymbol{J}'$$



No self-gravity →

Polyachenko & Shukhman (1982)
Chavanis (2012)

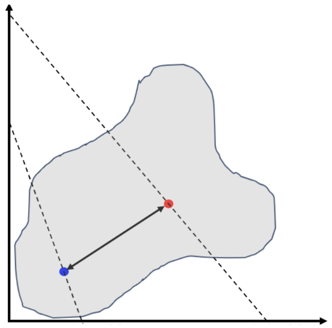
Landau (RR)

$$\sum_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})=\boldsymbol{k}'\cdot\boldsymbol{\Omega}(\boldsymbol{J}')$$

$$\psi_{\boldsymbol{k}\boldsymbol{k}'}$$

$$\int \mathrm{d}\boldsymbol{J}'$$



Local homogeneity →

Chandrasekhar (1943)
Chavanis (2013)

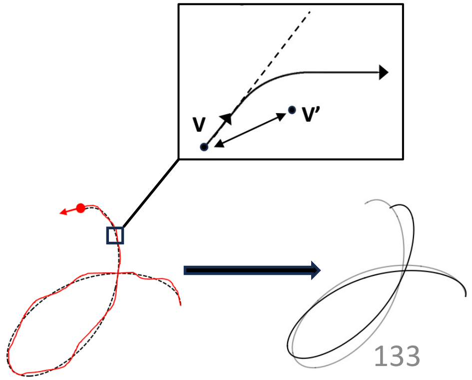
Orbit-averaged Chandrasekhar

$$\int \mathrm{d}\boldsymbol{k}$$

$$\boldsymbol{k}\cdot\boldsymbol{v}=\boldsymbol{k}\cdot\boldsymbol{v}'$$

$$\widehat{u}(\mathbf{k})$$

$$\int \mathrm{d}\boldsymbol{v}'$$

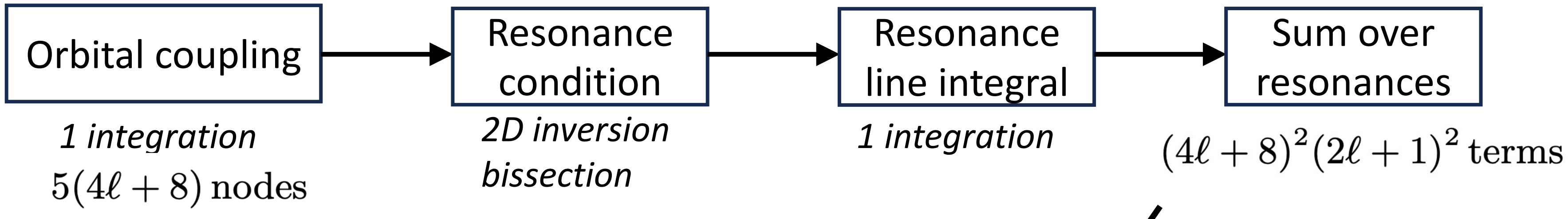


Landau equation

$$\frac{\partial F}{\partial t}(\boldsymbol{J},t) = \pi(2\pi)^3 \frac{M}{N} \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}' |\psi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2 \, \delta_{\mathrm{D}}(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}') \times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) F(\boldsymbol{J},t)F(\boldsymbol{J}',t),$$

$|\psi_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}\cdot\boldsymbol{\Omega})|^2$ **Bare** orbital coupling

Landau equation



$\ell = 0 : 64$ terms

$\ell = 1 : 1296$ terms

$\ell = 2 : 6400$ terms

$\ell = 5 : 94\,864$ terms

$\ell = 10 : 1\,016\,064$ terms

$\ell = 20 : 13\,017\,664$ terms

$\ell = 50 : 441\,336\,064$ terms

Landau prediction

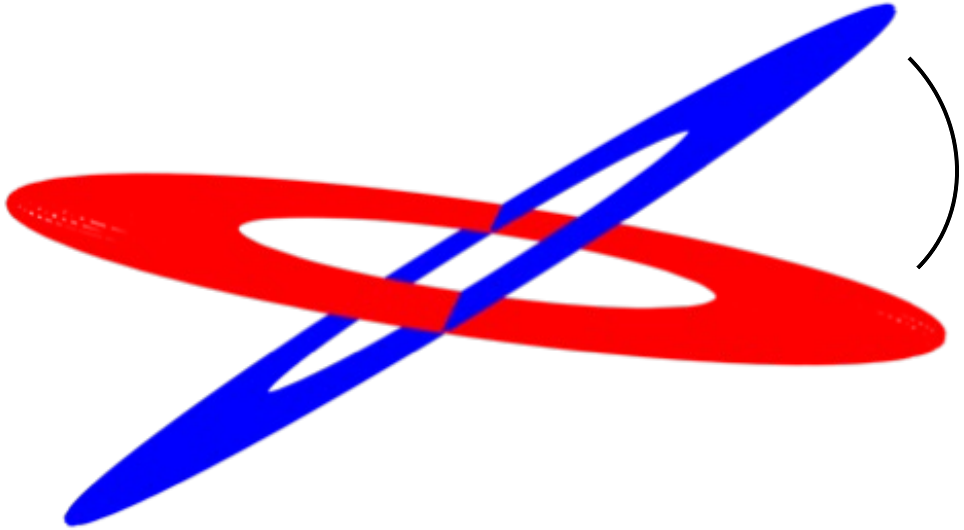
$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{J}) = -\sum_{\ell=0}^{\infty} \frac{\partial}{\partial \boldsymbol{J}} \cdot \boldsymbol{\mathcal{F}}_{\ell}(\boldsymbol{J})$$

Hamilton et al. (2018)

Landau prediction

Harmonic decomposition of the spherical potential

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{J}) = -\sum_{\ell=0}^{\infty} \frac{\partial}{\partial \boldsymbol{J}} \cdot \boldsymbol{\mathcal{F}}_{\ell}(\boldsymbol{J})$$



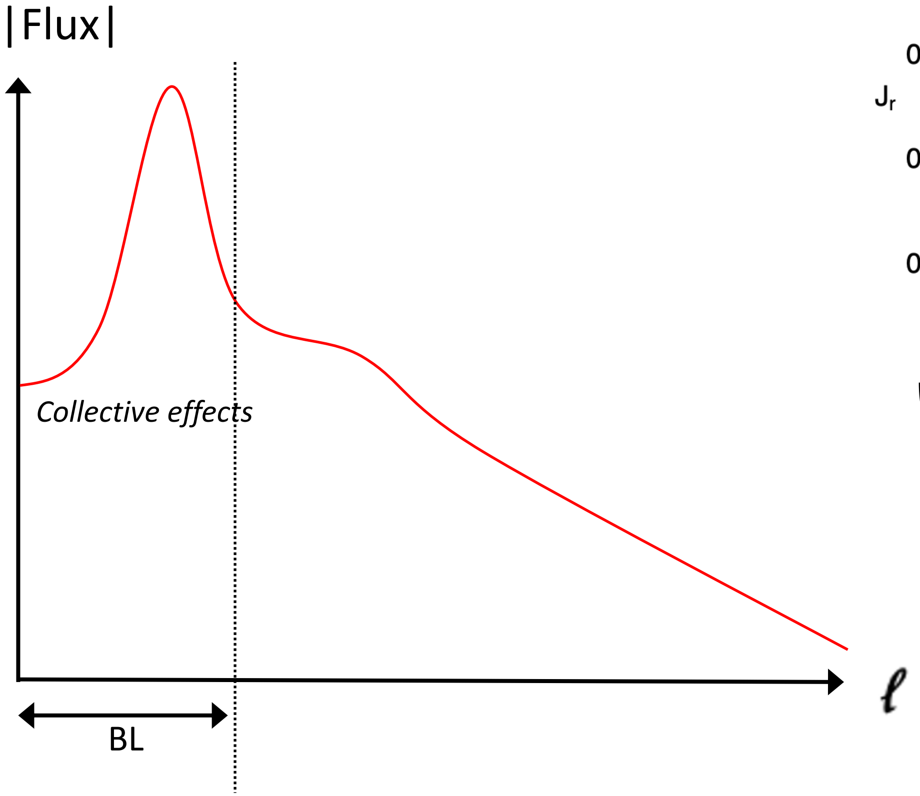
$\sim \frac{\pi}{\ell}$

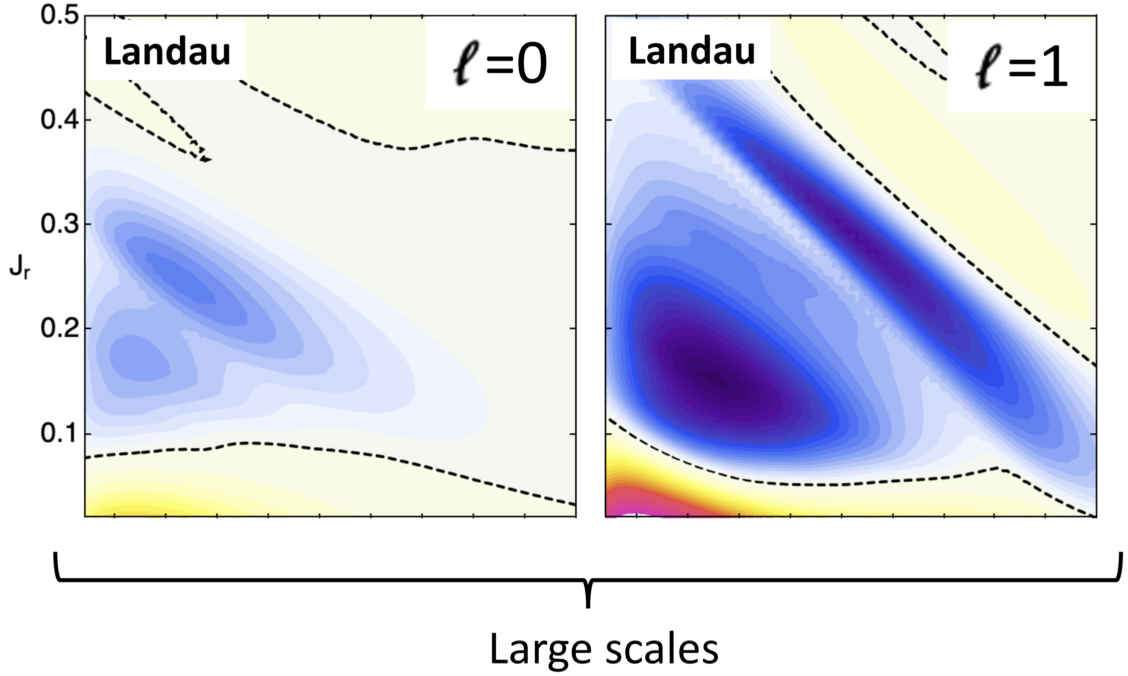
→ Decompose interactions w.r.t. relative orbital planes

Hamilton et al. (2018)

Impact of resonances

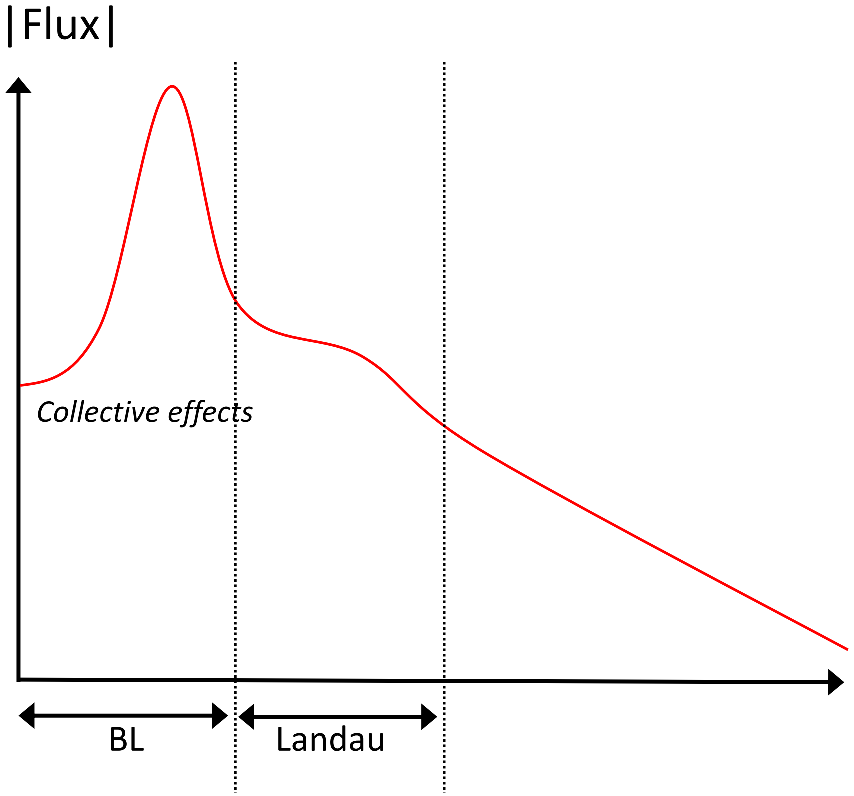
- Scale separation

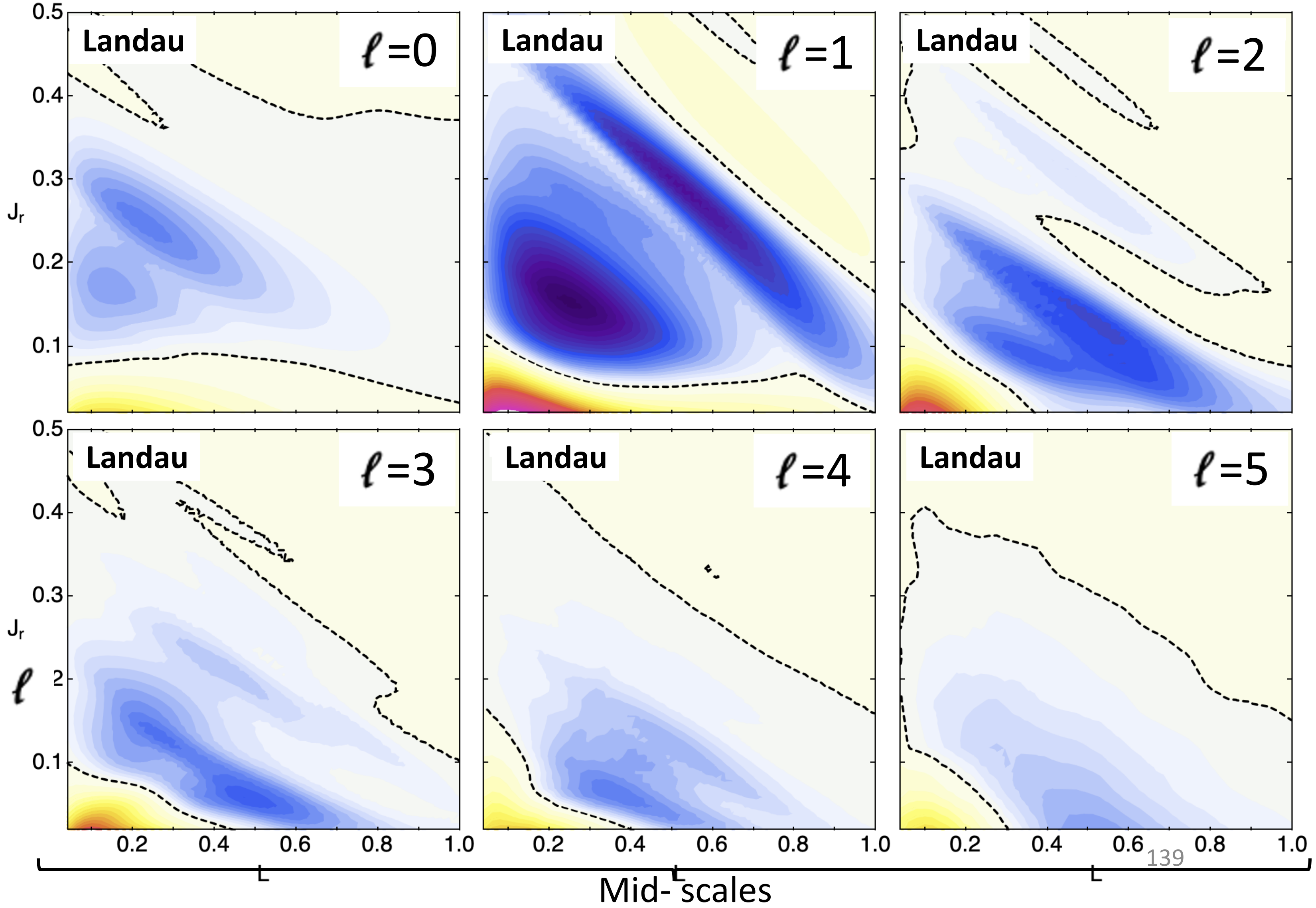




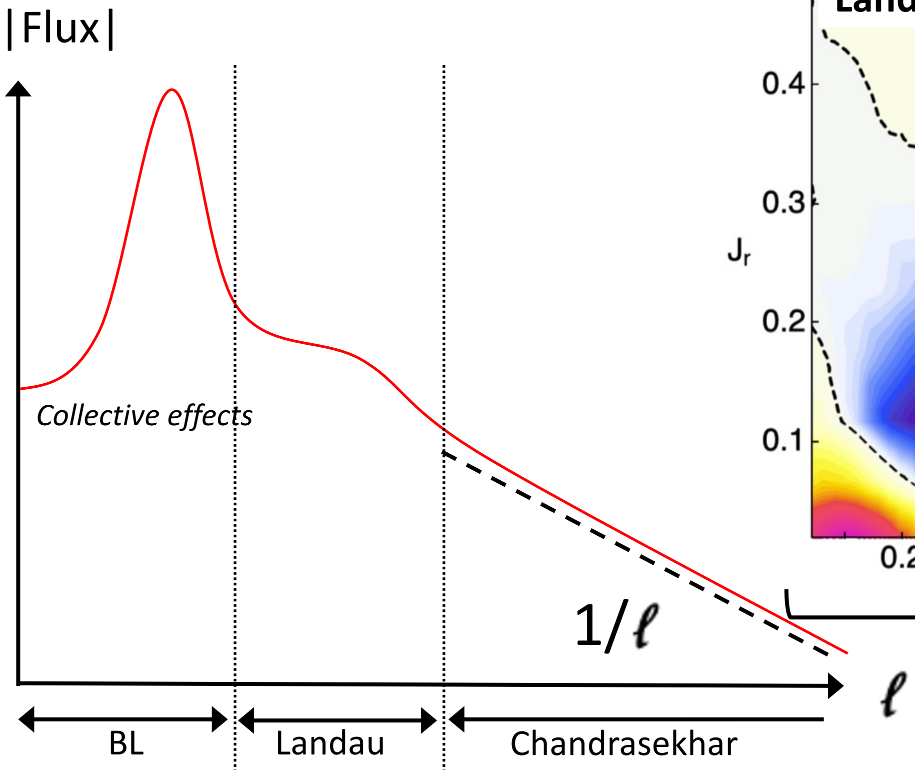
Impact of resonances

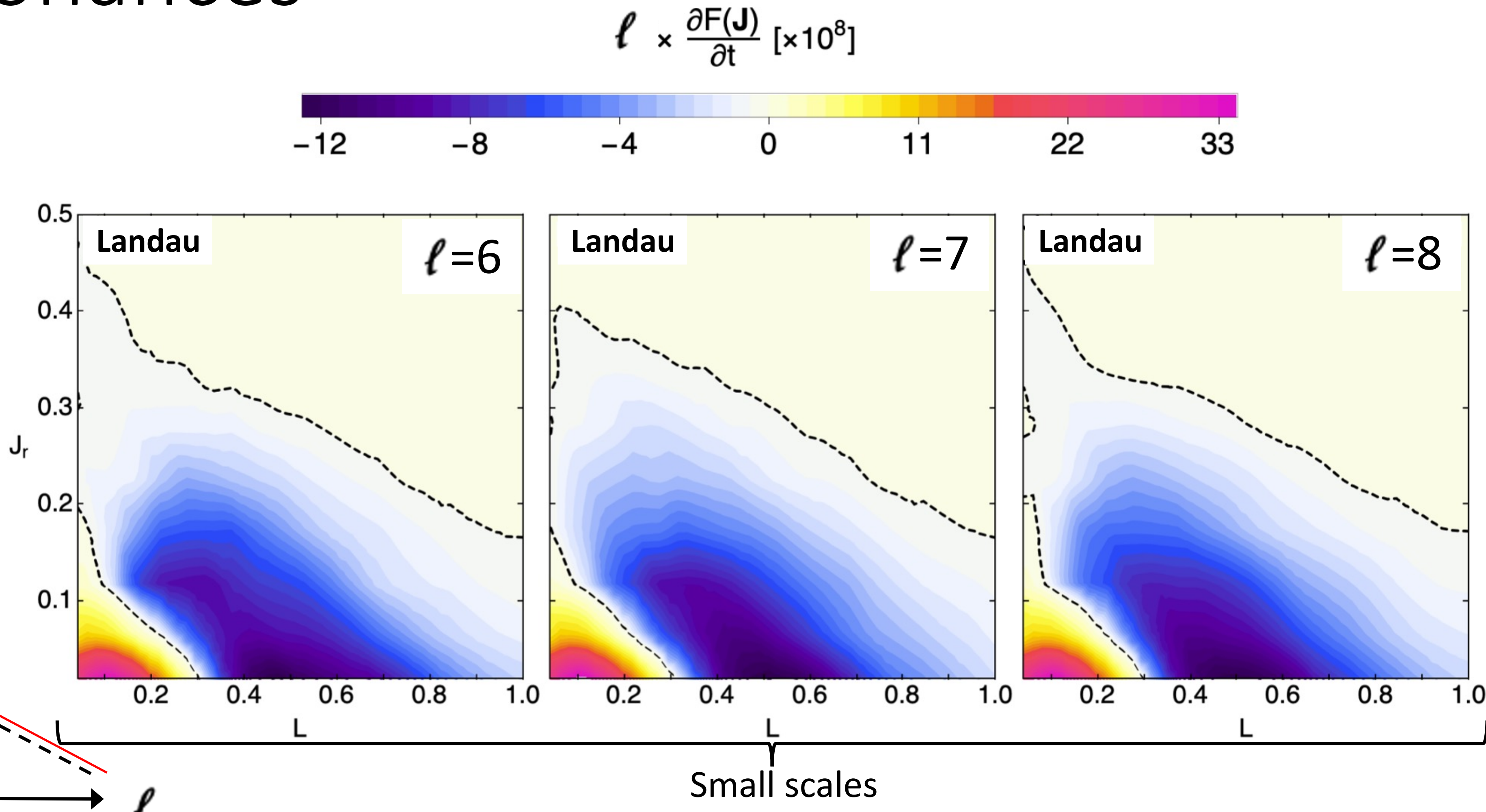
- Scale separation



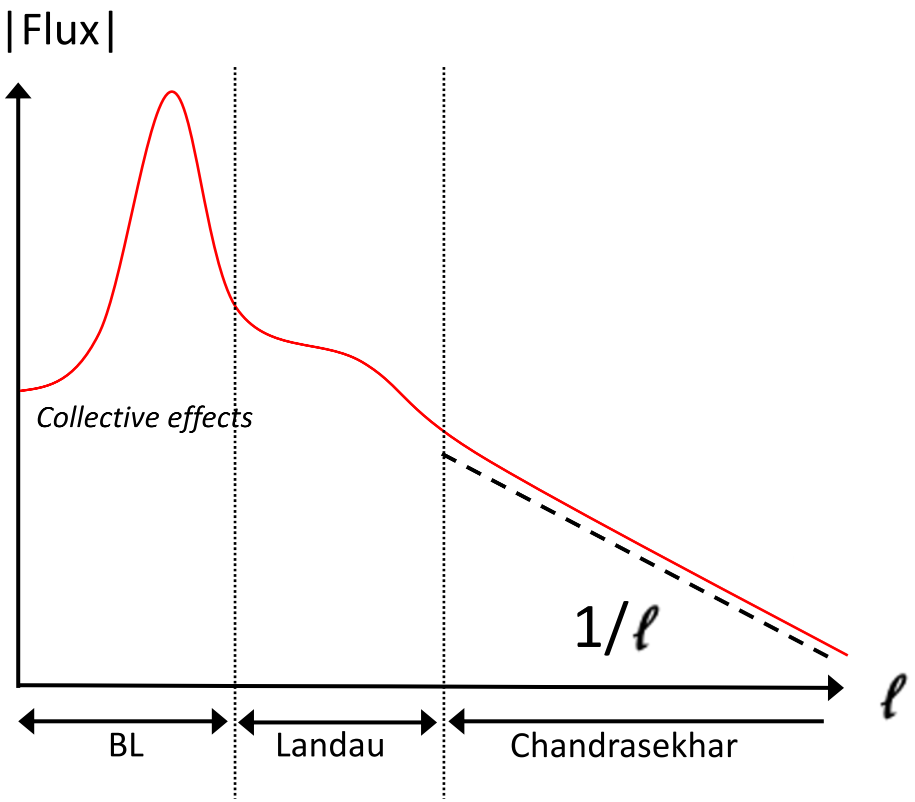


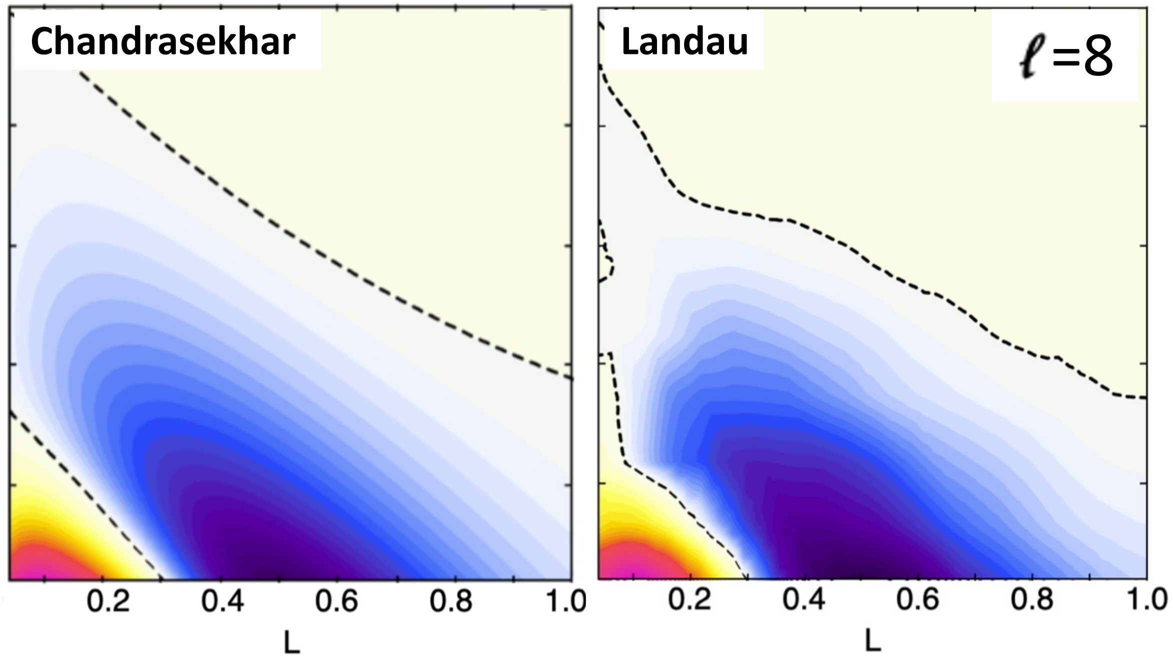
Impact of resonances





Impact of resonances





High harmonics : Chandrasekhar theory
What about small harmonics ?

What about rotation?

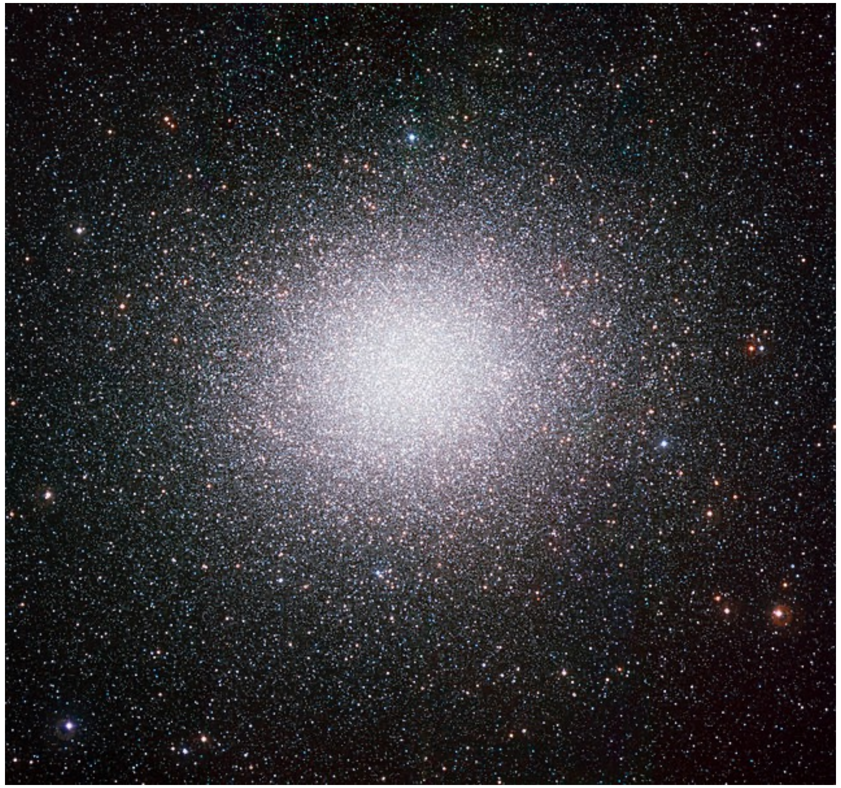
Credits: *WFI camera, ESO's La Silla Observatory*

- How to make theoretical predictions ?
- What mechanisms impact secular evolution?
- **How does kinematics impact evolution ?**

ω Cen

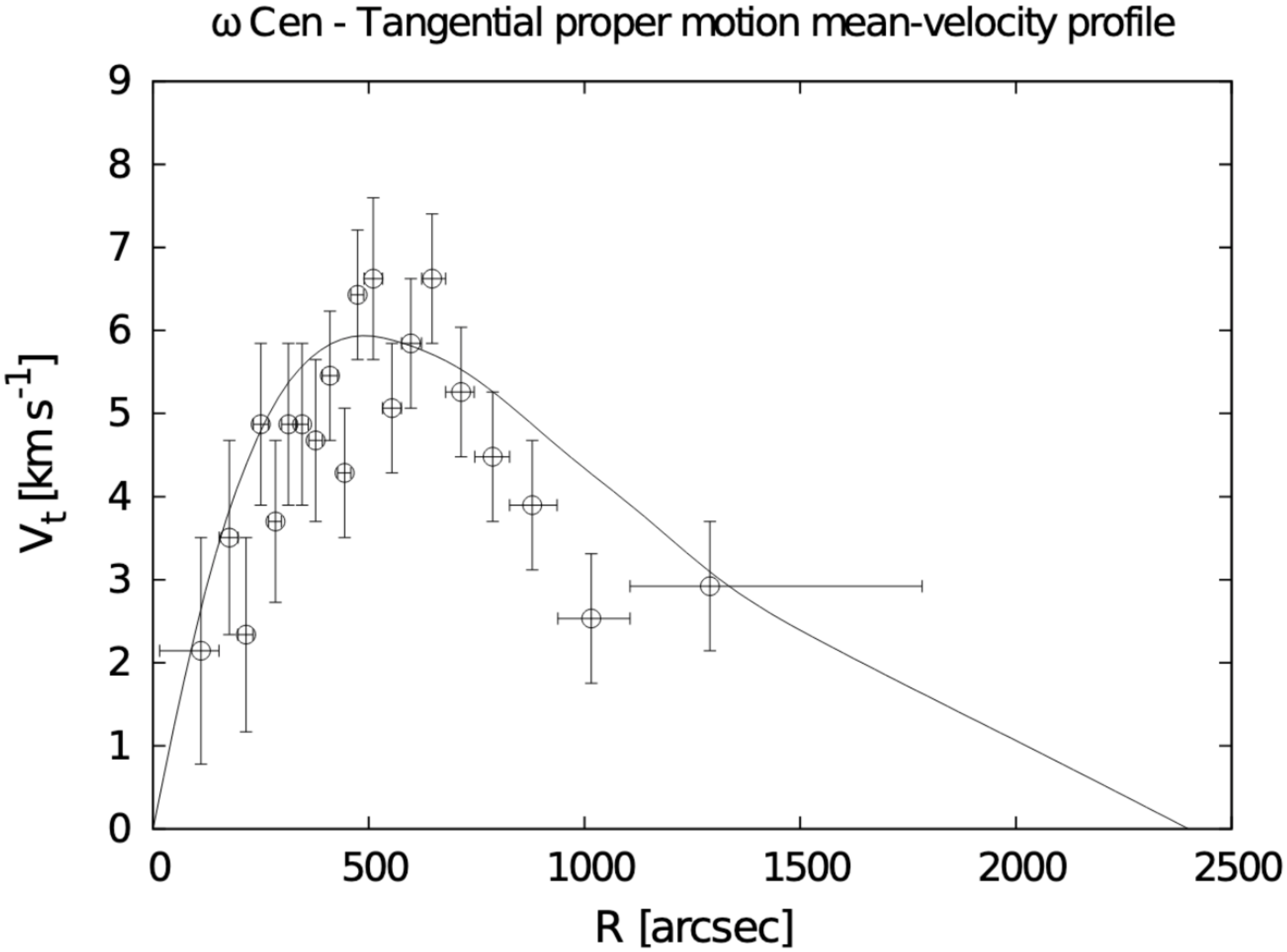
Impact of rotation

- Rotation curve



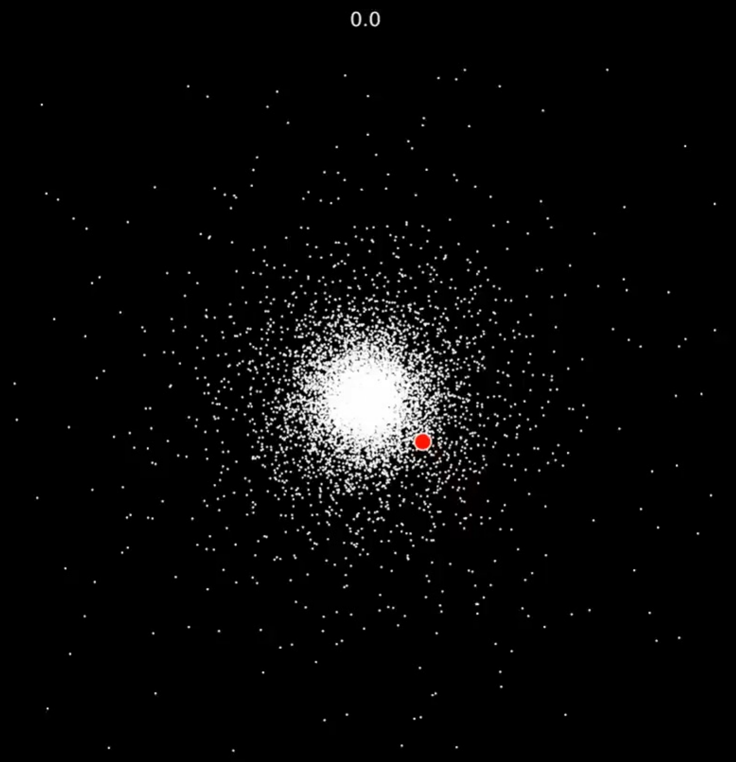
ω Cen

Credits: *WFI camera, ESO's La Silla Observatory*

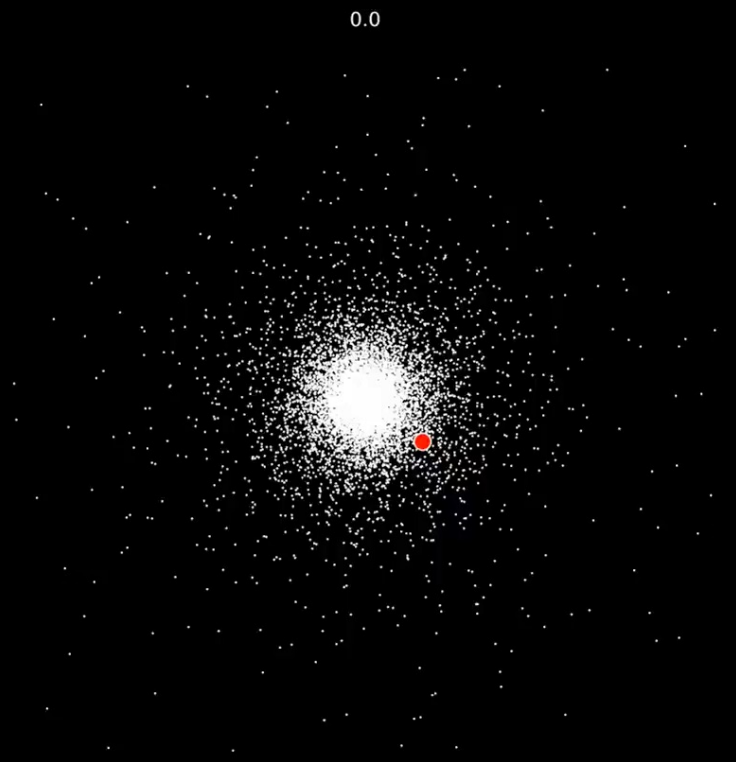


The rotating Plummer cluster

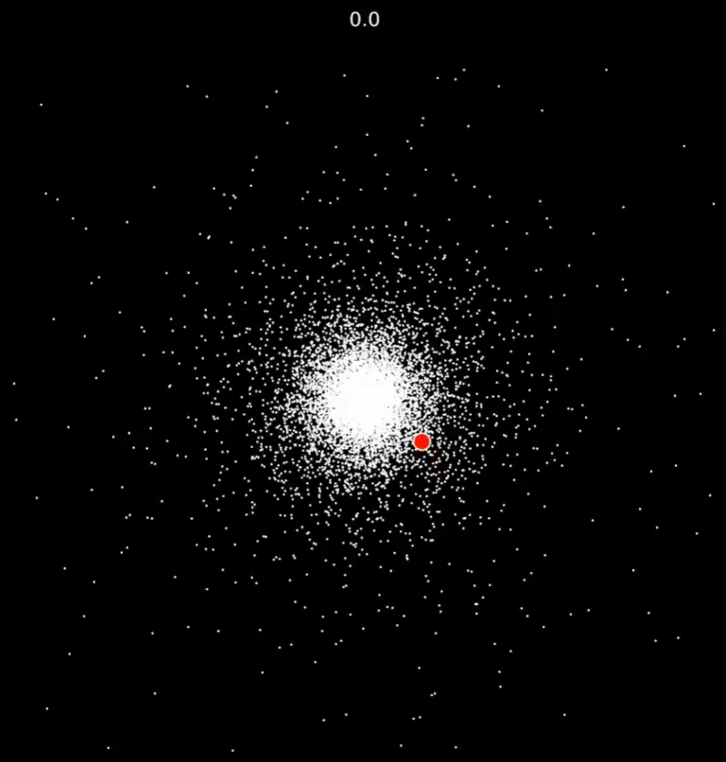
No rotation α=0



Rotation α=0.25



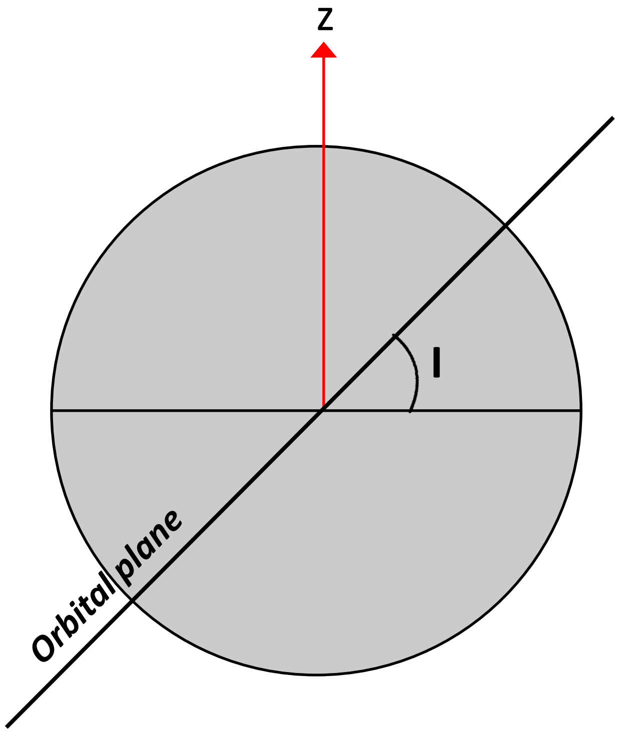
Rotation α=0.5



Z-axis towards us

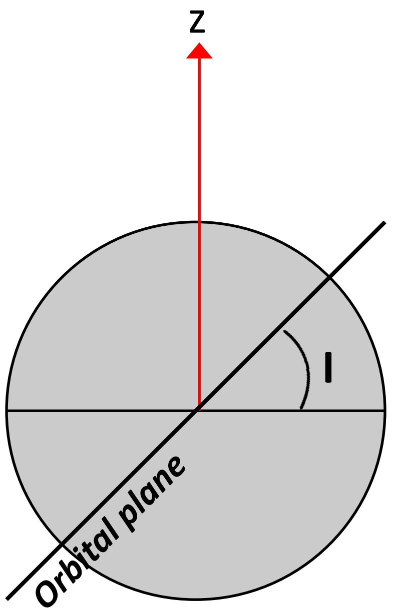
The rotating Plummer cluster

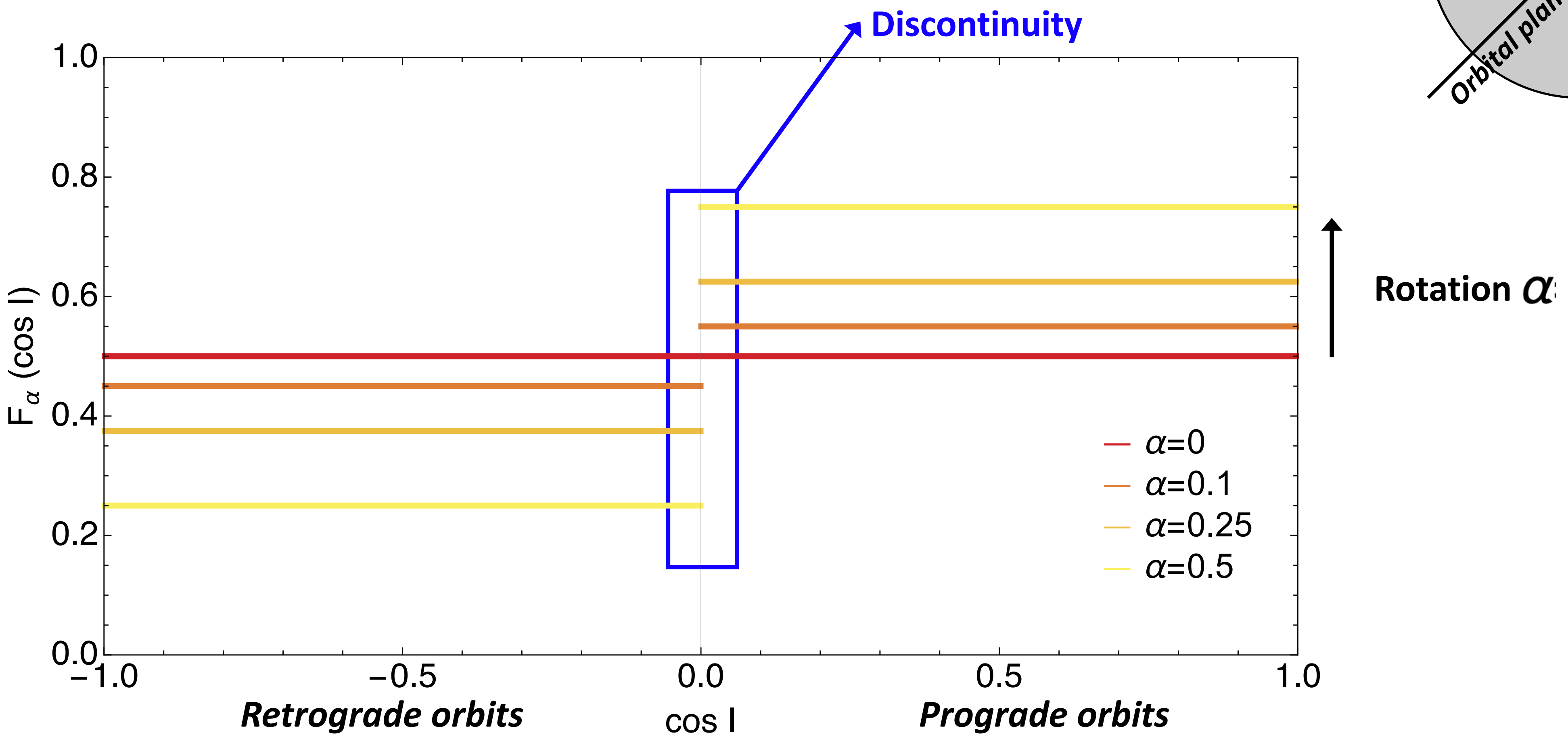
- Preferential axis: rotation around (Oz)
- Orbital inclination I: $\cos I = L_z / L$
- 3D action space



The rotating Plummer cluster

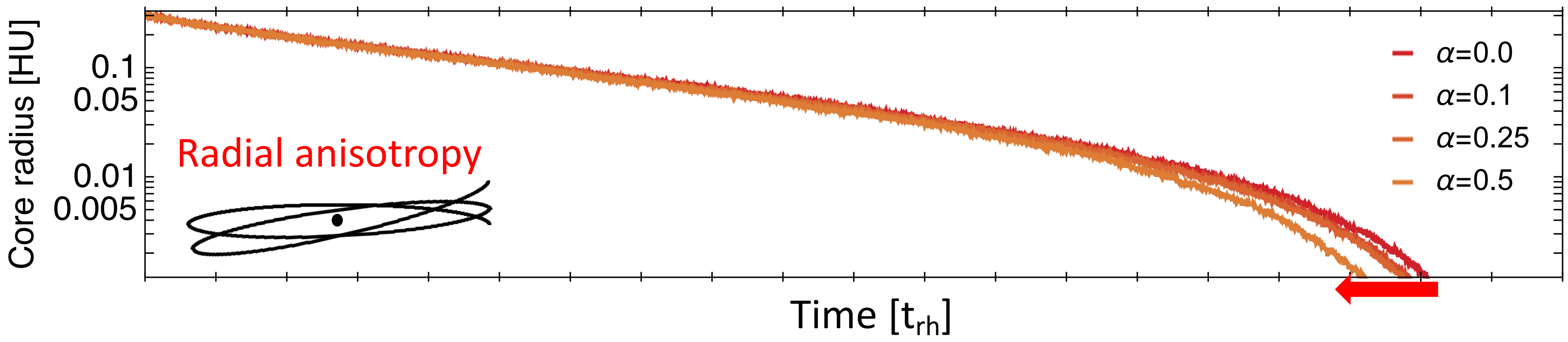
- Anisotropic Plummer cluster
- Lynden-Bell demon: preserves spherical symmetry and mean field





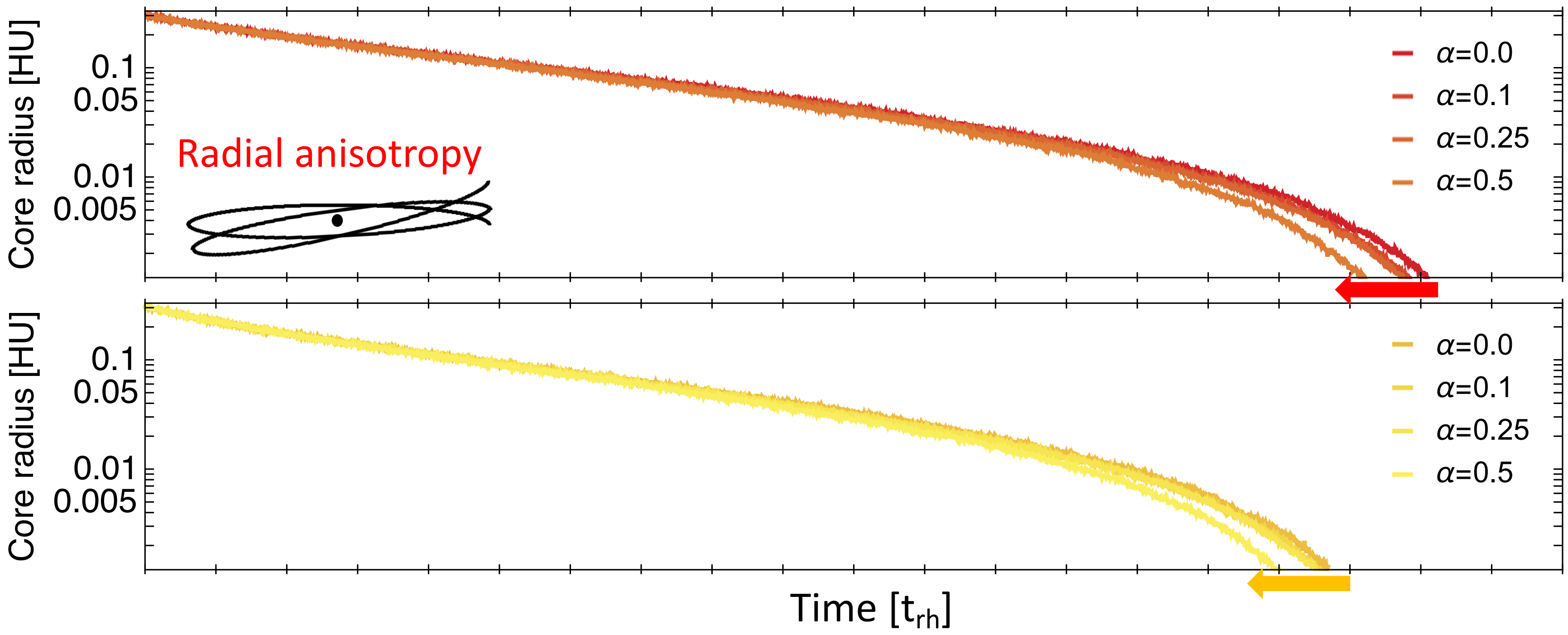
Gravo gyro catastrophe?

- Numerical simulation: average over 50 realisations



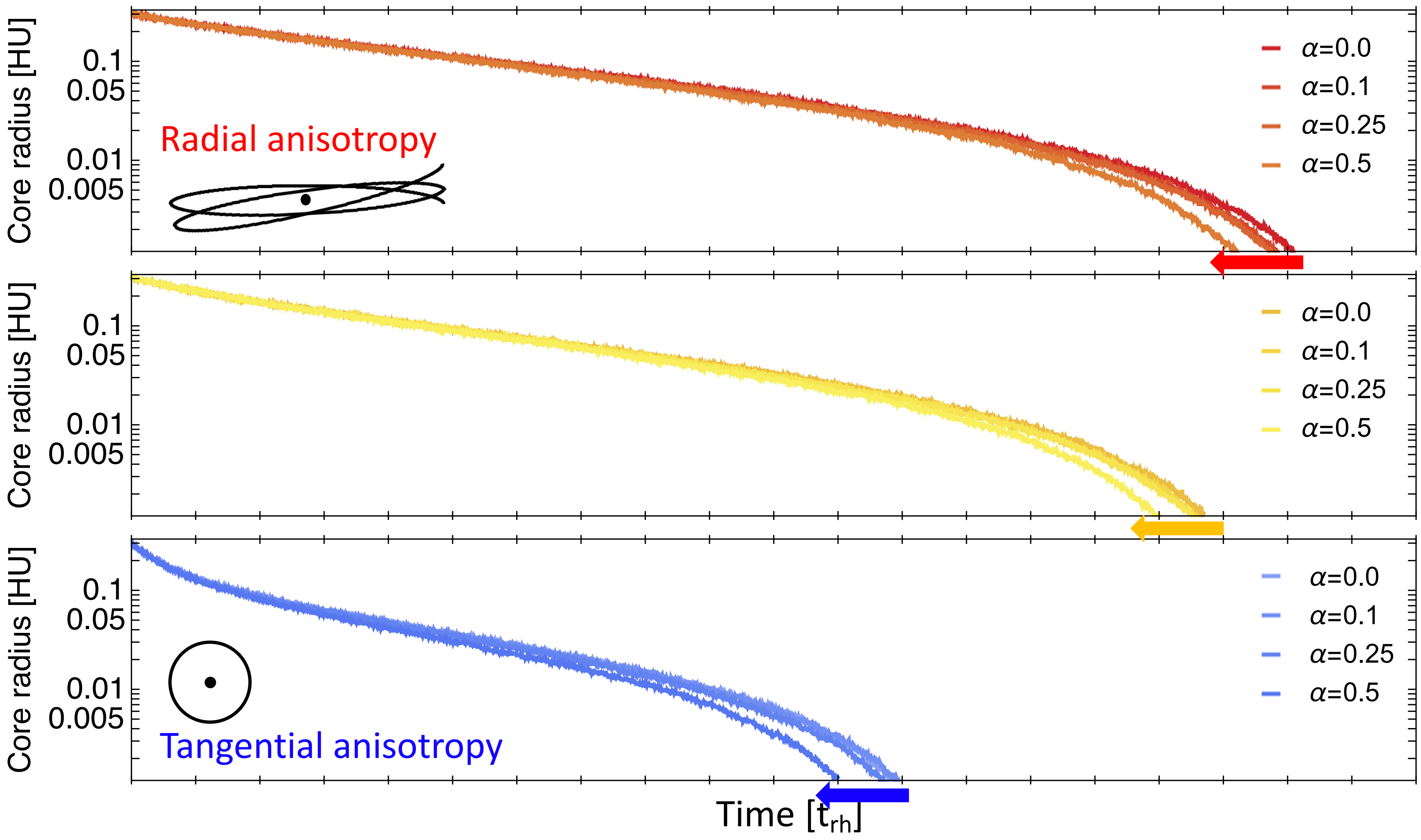
Gravo gyro catastrophe?

- Numerical simulation: average over 50 realisations



Gravo gyro catastrophe?

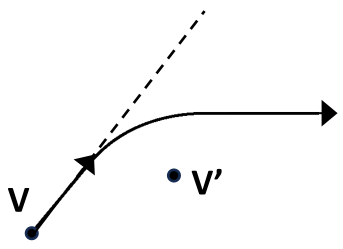
- Numerical simulation: average over 50 realisations

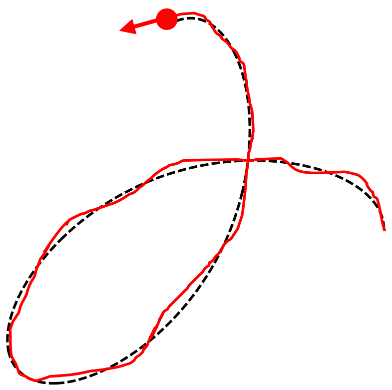


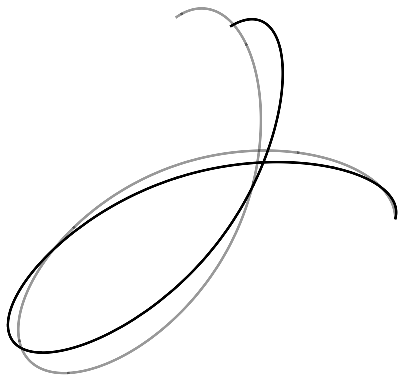
Theoretical prediction: Chandrasekhar theory

Velocity deflections → (2^{nd} order variation) → Motion integral diffusion → (Orbit-average) → Orbit-averaged diffusion coefficients

- **Velocity deflections:** *3 integrations*, *1 additional parameter*
- **Motion integral diffusion:** *2D inversion*
- **Orbit-averaged diffusion coefficients:** *2 integrations*, *3D sampling*

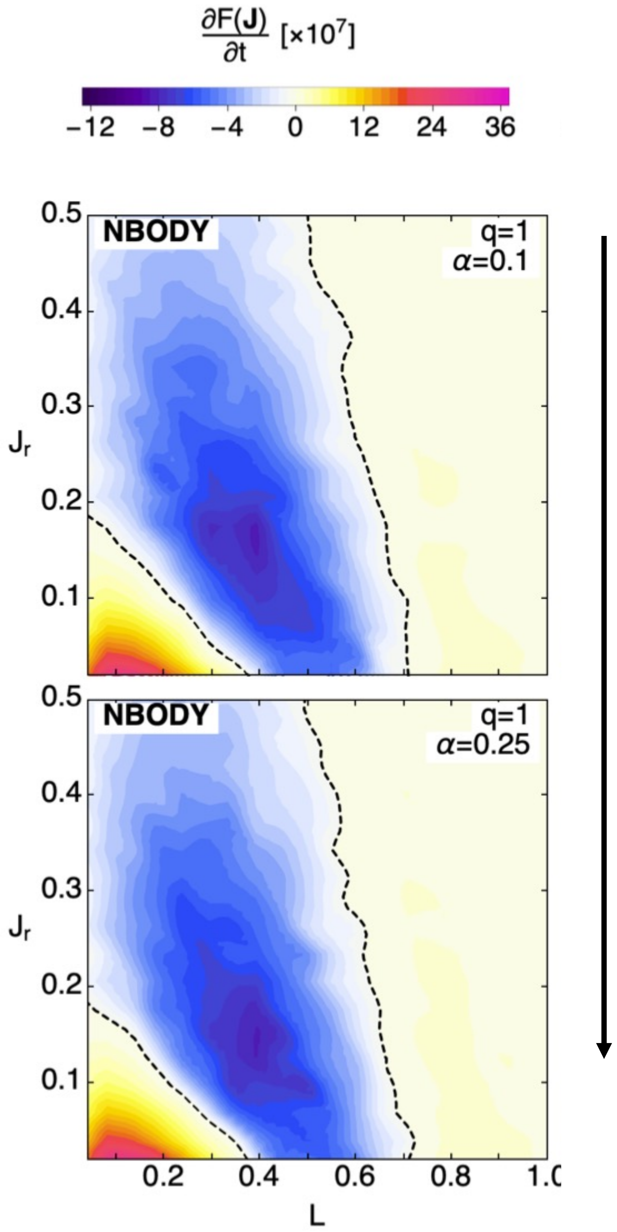






(L, J_r)-space

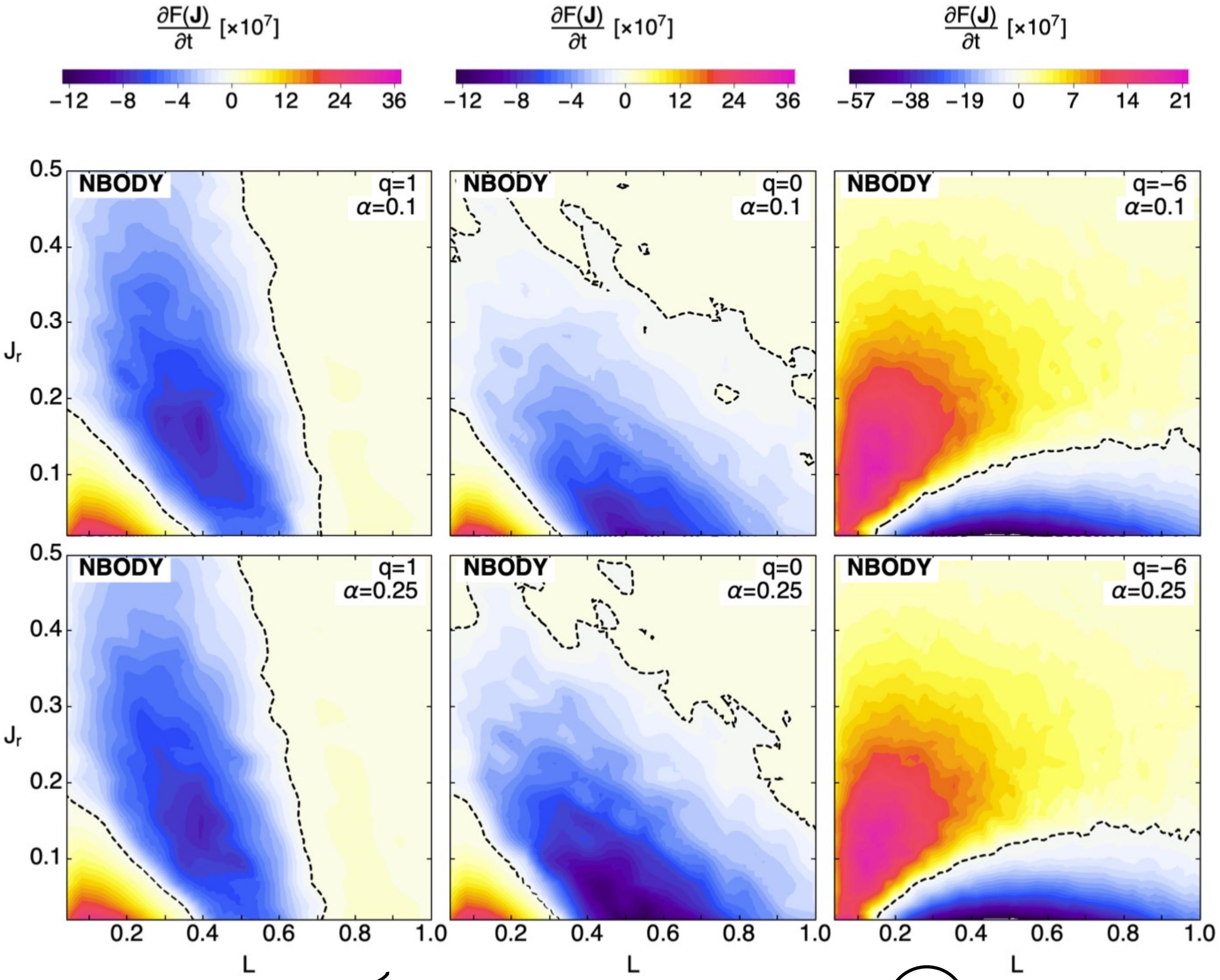
- N-body measurement
- Relaxation rate: $dF/dt_{|t=0^+}$

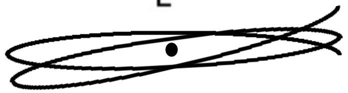


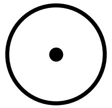
Small impact of rotation on relaxation rate

(L, J_r)-space

- N-body measurement
- Relaxation rate: $dF/dt_{|t=0^+}$

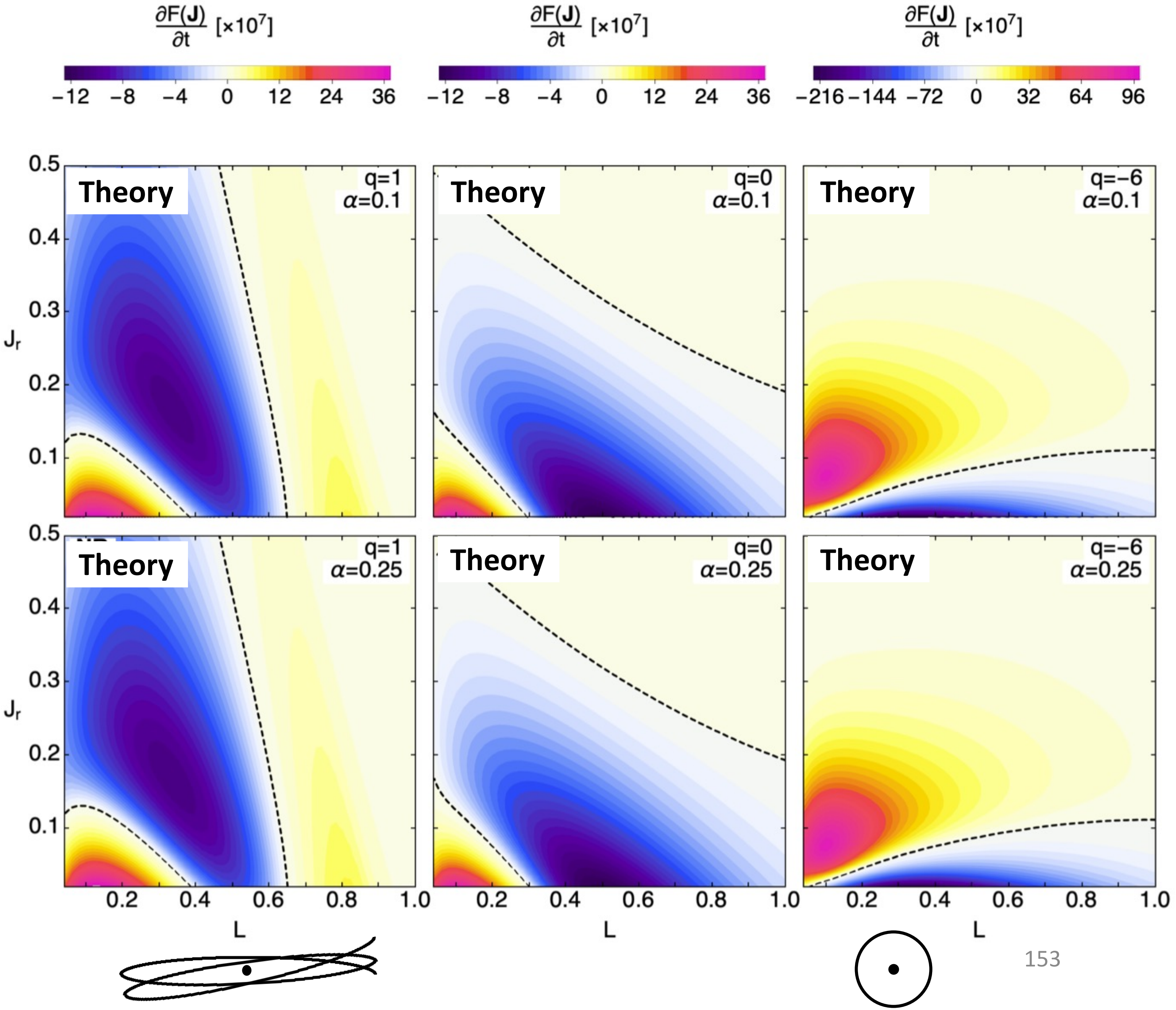






(L, J_r)-space

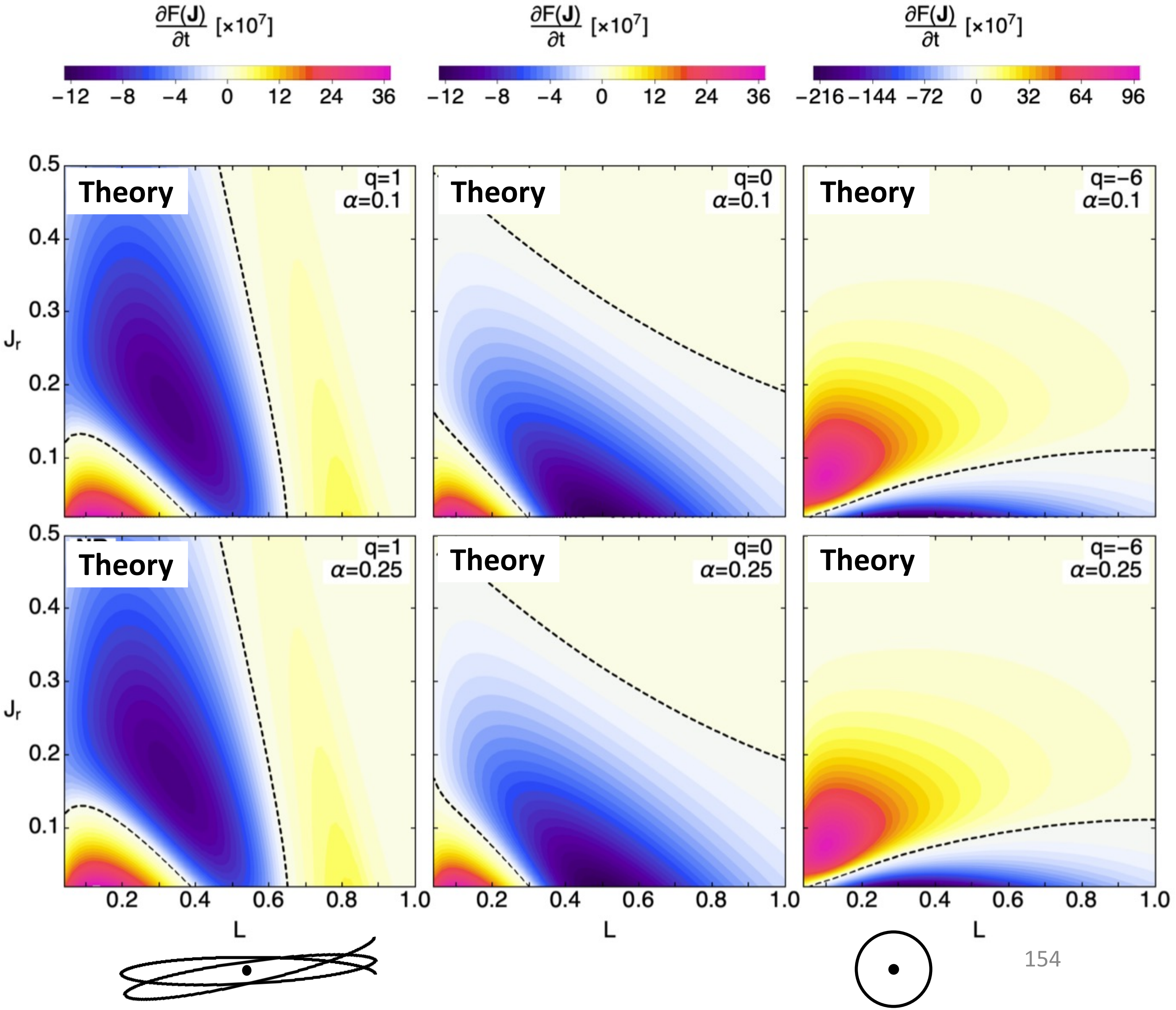
- Theoretical prediction
- Relaxation rate: $dF/dt_{|t=0^+}$



(L, J_r)-space

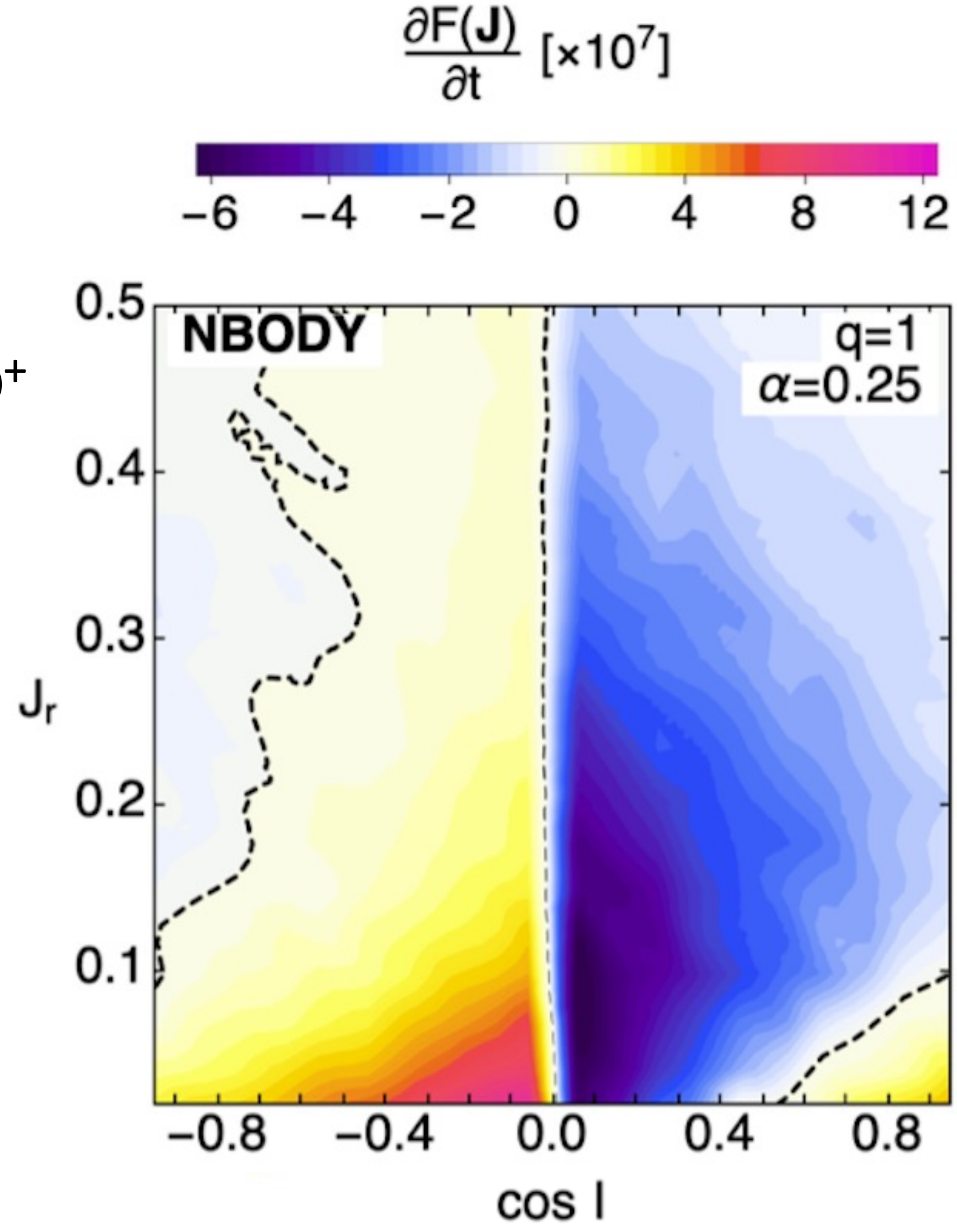
- Theoretical prediction
- Relaxation rate: $dF/dt_{|t=0^+}$

Satisfying prediction



($\cos I$, J_r)-space

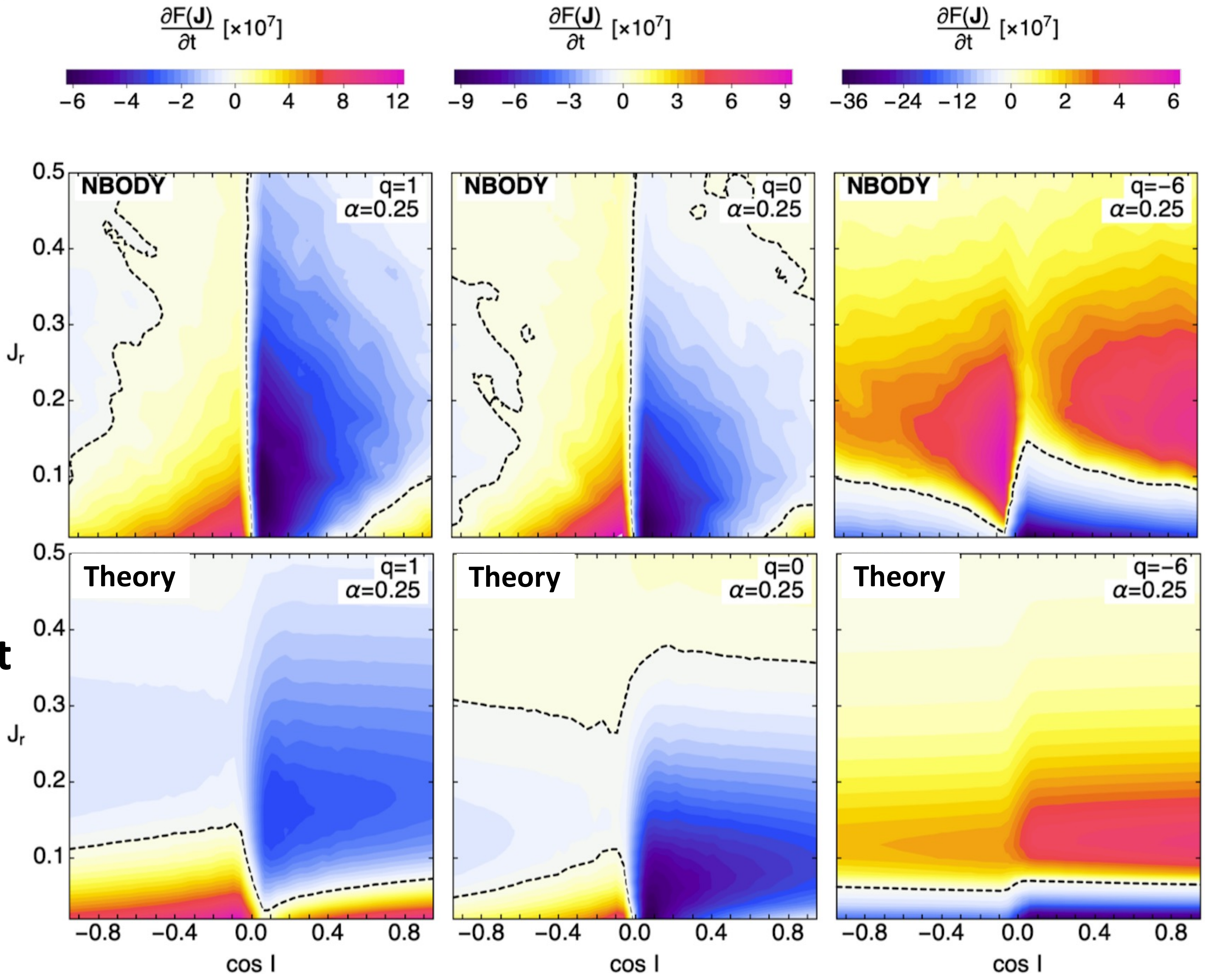
- N-body measurement
- Relaxation rate: $dF/dt_{|t=0^+}$



Reduction of discontinuities

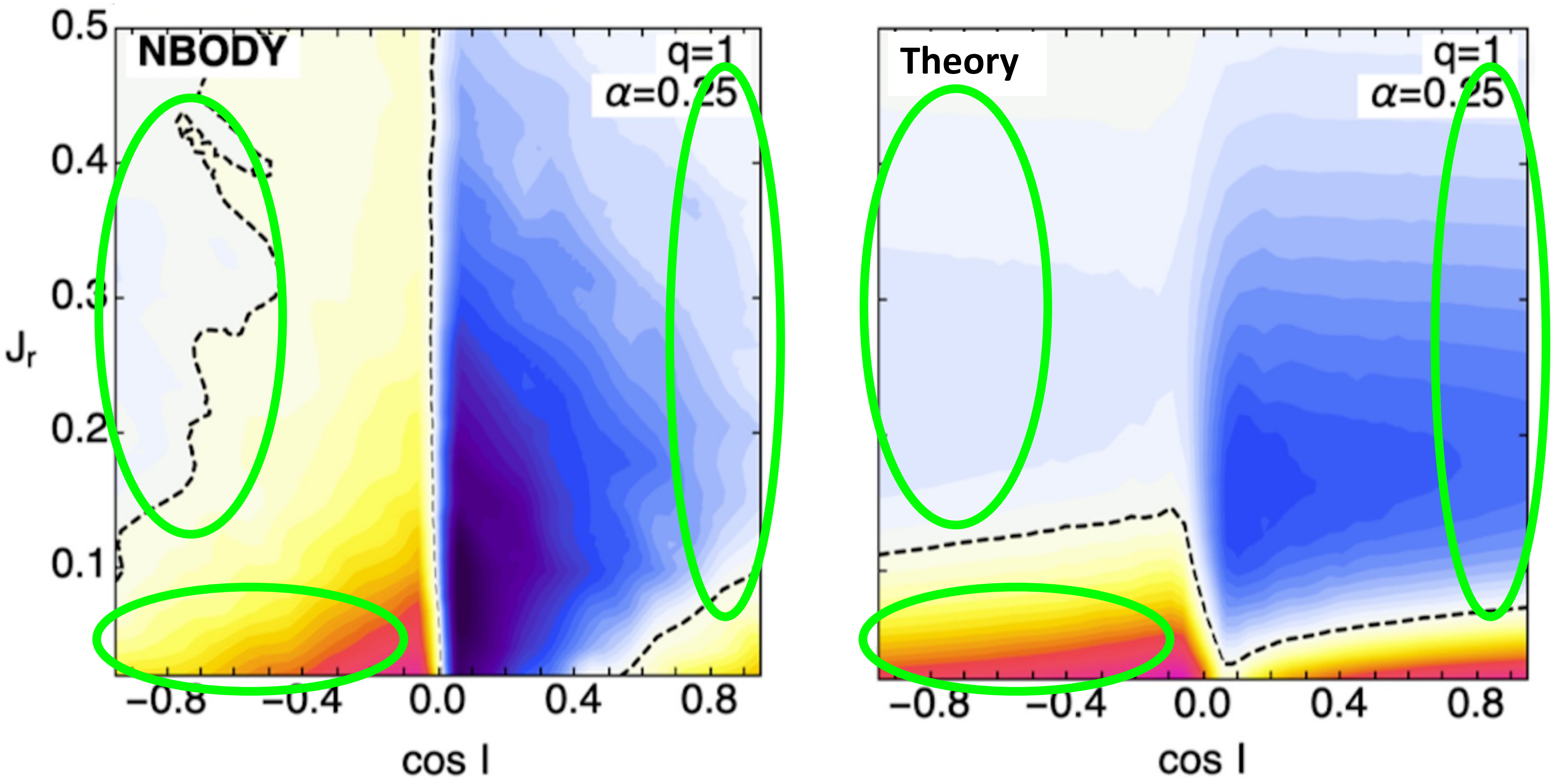
(cos I, J_r)-space

- Relaxation rate: $dF/dt_{|t=0}$



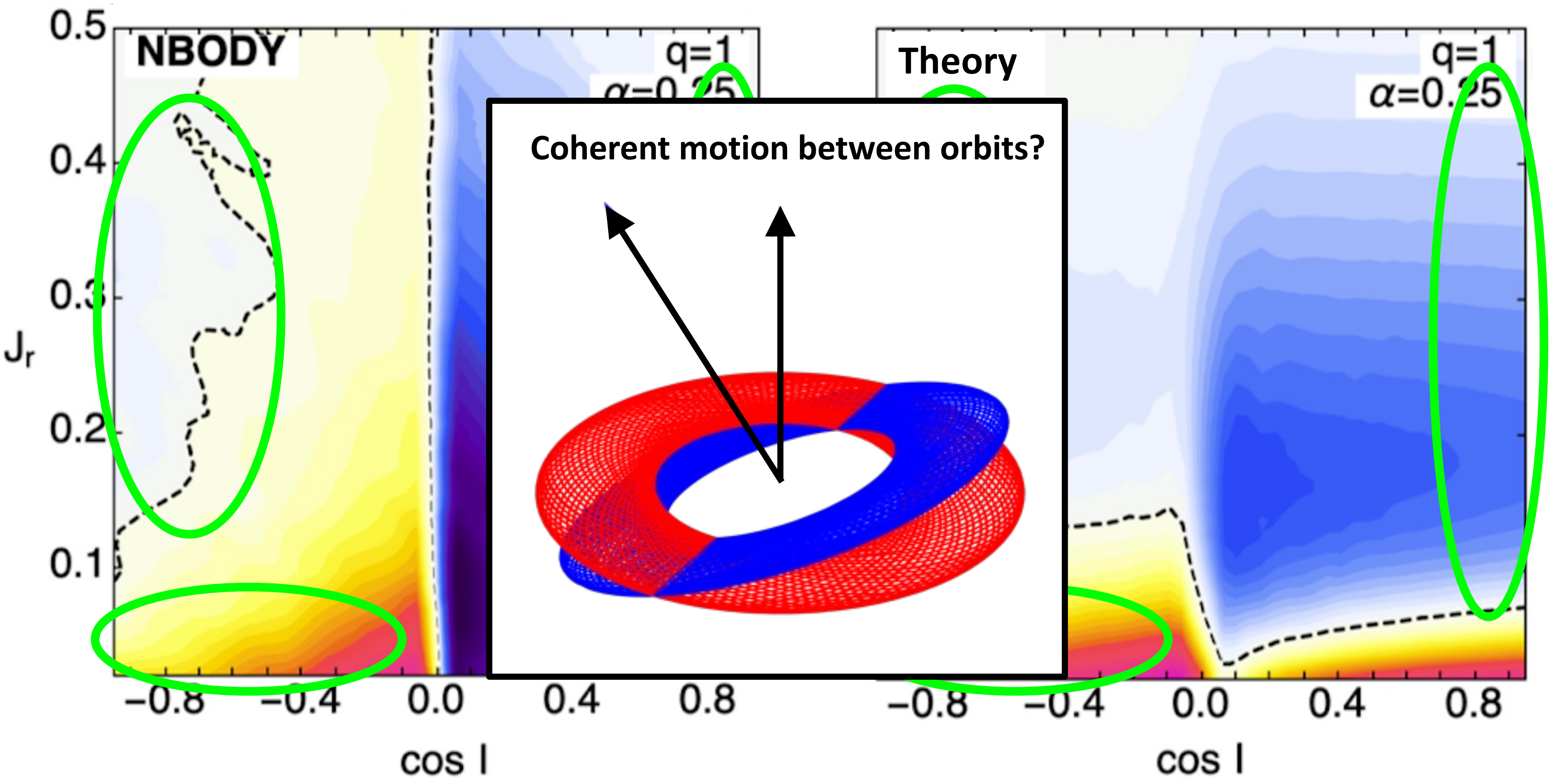
Qualitative agreement

(cos I, J_r)-space



Discrepancies

(cos I, J_r)-space

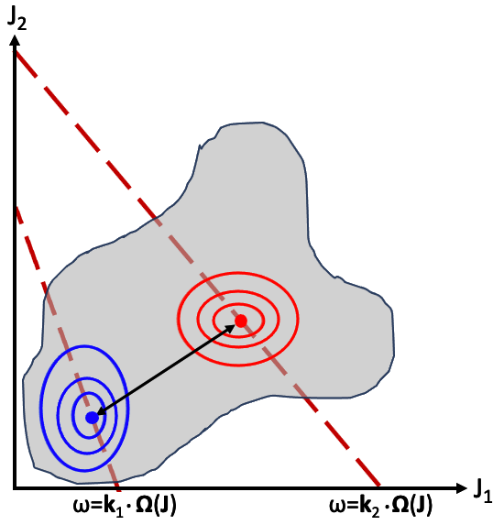


Discrepancies

Conclusions

How can I make theoretical predictions ?

Balescu-Lenard, Landau, Chandrasekhar



Conclusions

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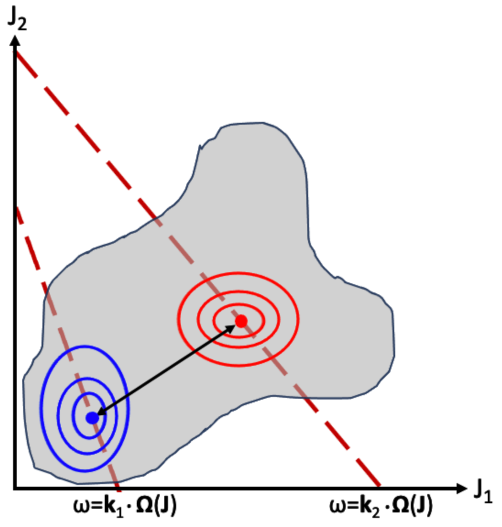
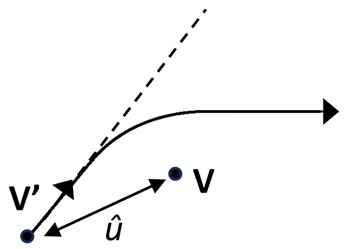
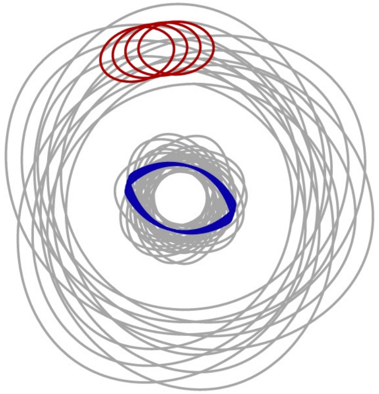


What mechanisms impact secular evolution?

Pairwise deflections, coherent interactions



2-body deflections

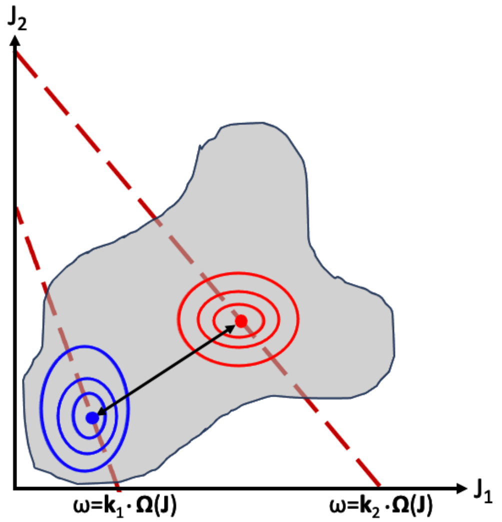


Coherent interactions

Conclusions

How can I make theoretical predictions ?

Balescu-Lenard, Landau, Chandrasekhar



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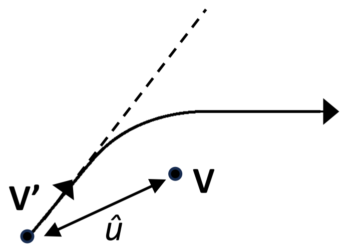
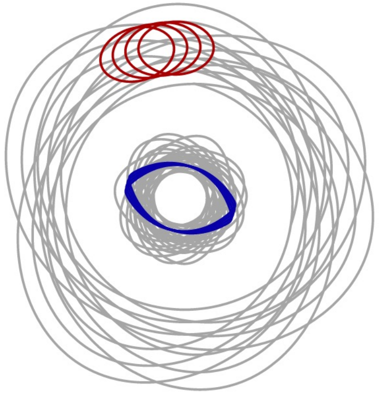


2-body deflections

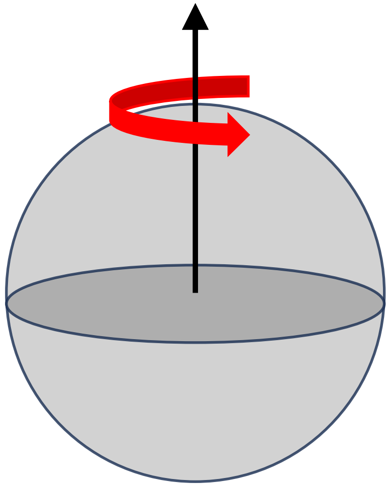


Coherent interactions

What are the origins of the differences in secular evolution?

Kinematic diversity

Anisotropy

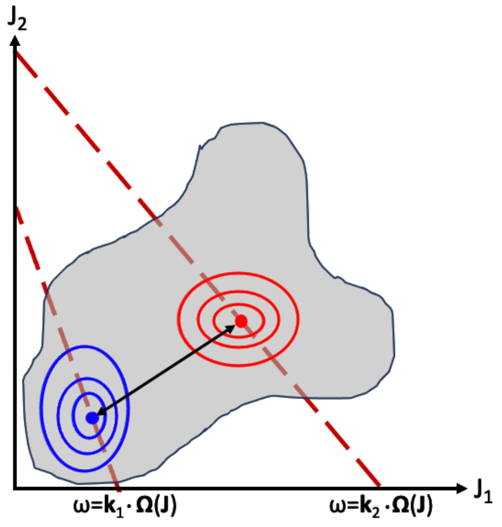


Rotation

Conclusions

How can I make theoretical predictions ?

Balescu-Lenard, Landau, Chandrasekhar



Tep et al. (2021)

Reddish, …, Tep et al. (2022)

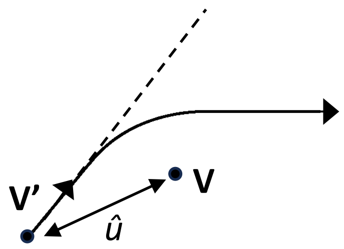
Astro

Tep et al. (2022)

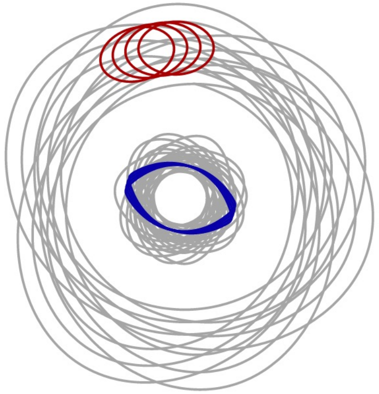
Theory

What mechanisms impact secular evolution?

Pairwise deflections, coherent interactions



2-body deflections



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What are the origins of the differences in secular evolution?

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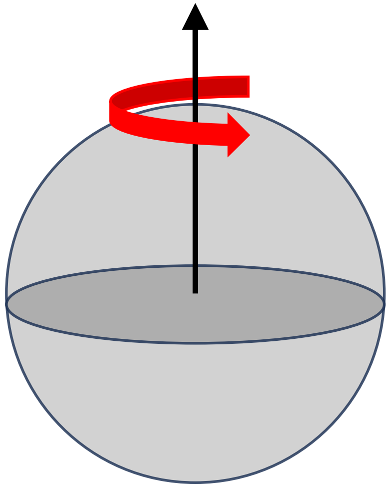
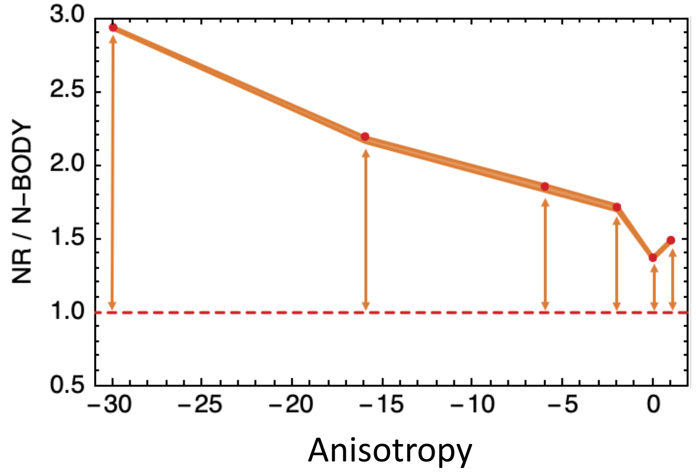
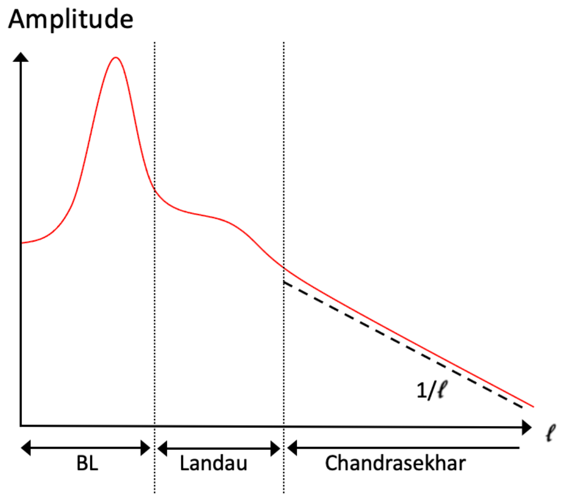


Rotation

Upcoming works

Coulomb logarithm

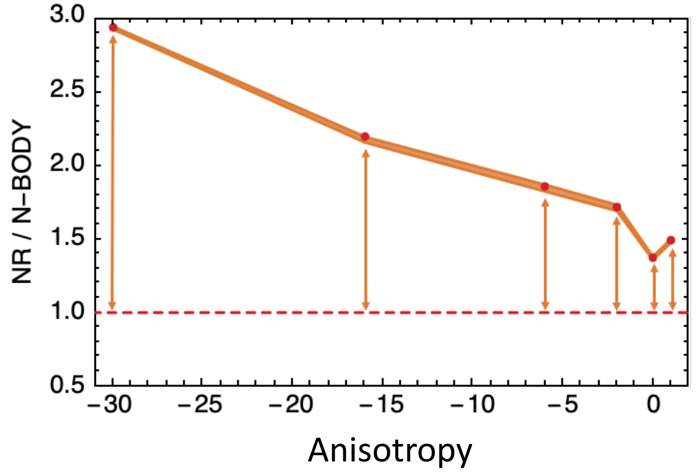


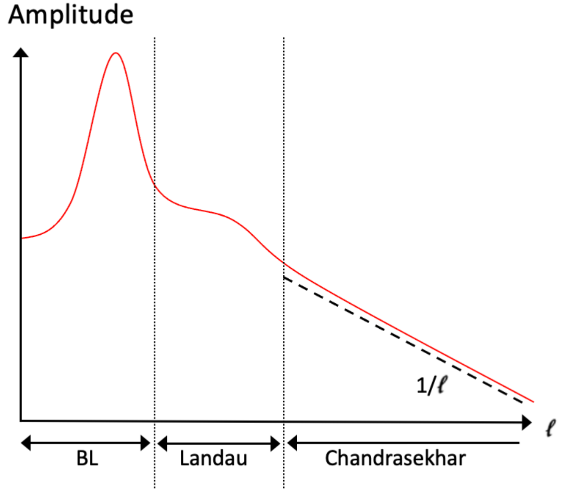


Heggie & Retterer

Upcoming works

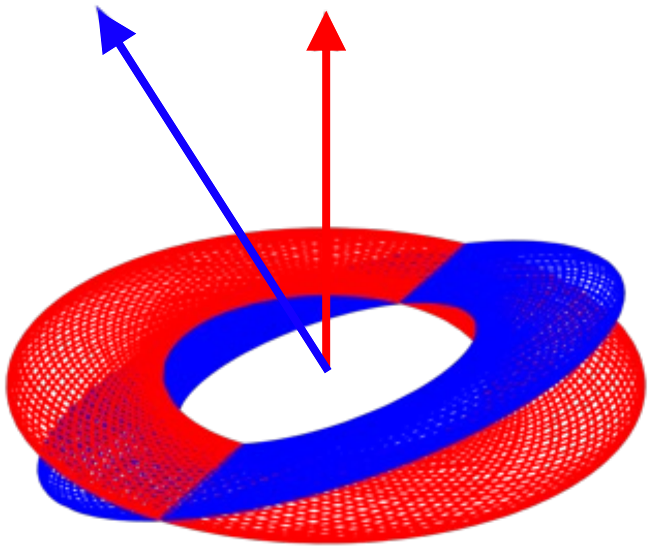
Coulomb logarithm





Heggie & Retterer

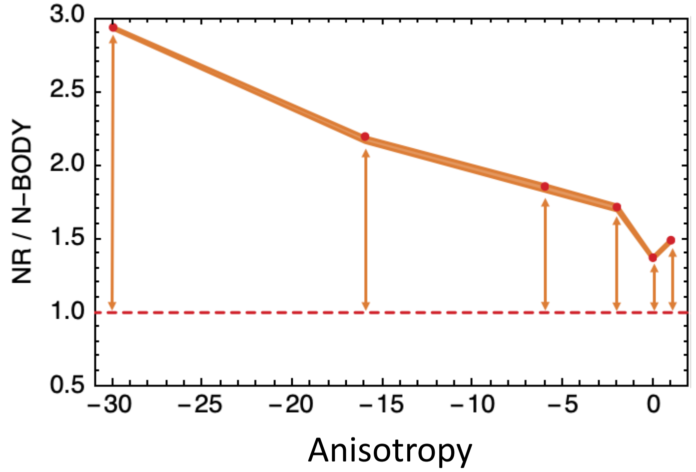
Vector resonant relaxation

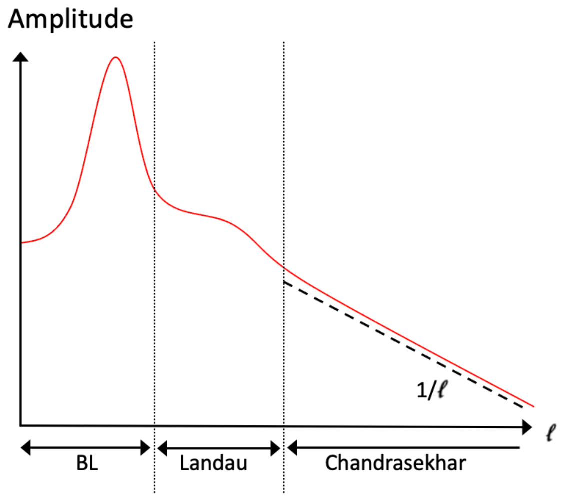


Kocsis & Tremaine (2011)
Szolgyen & Kocsis (2018)
Meiron & Kocsis (2019)

Upcoming works

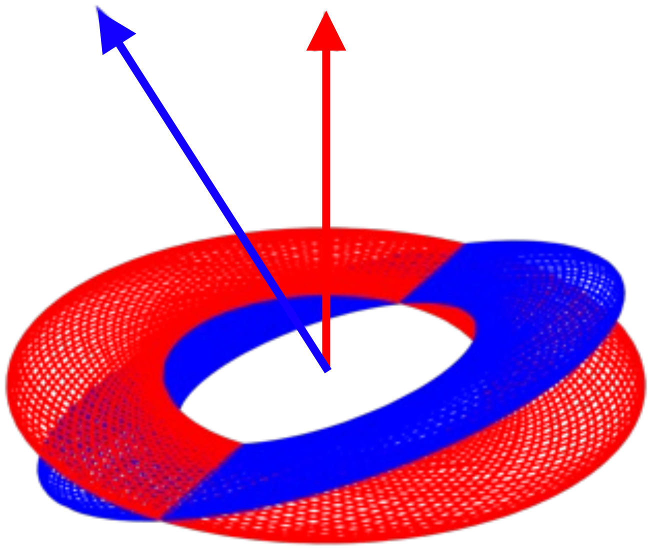
Coulomb logarithm





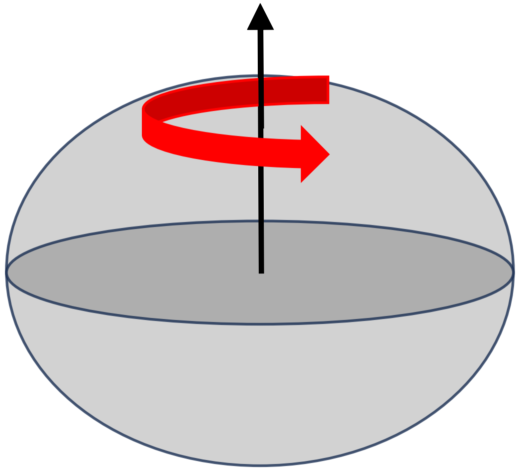
Heggie & Retterer

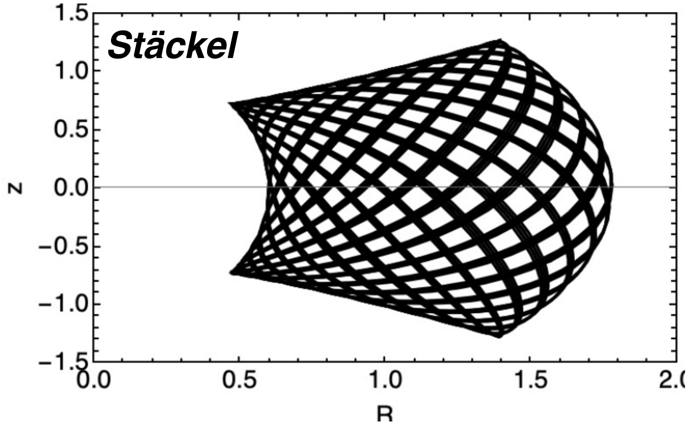
Vector resonant relaxation



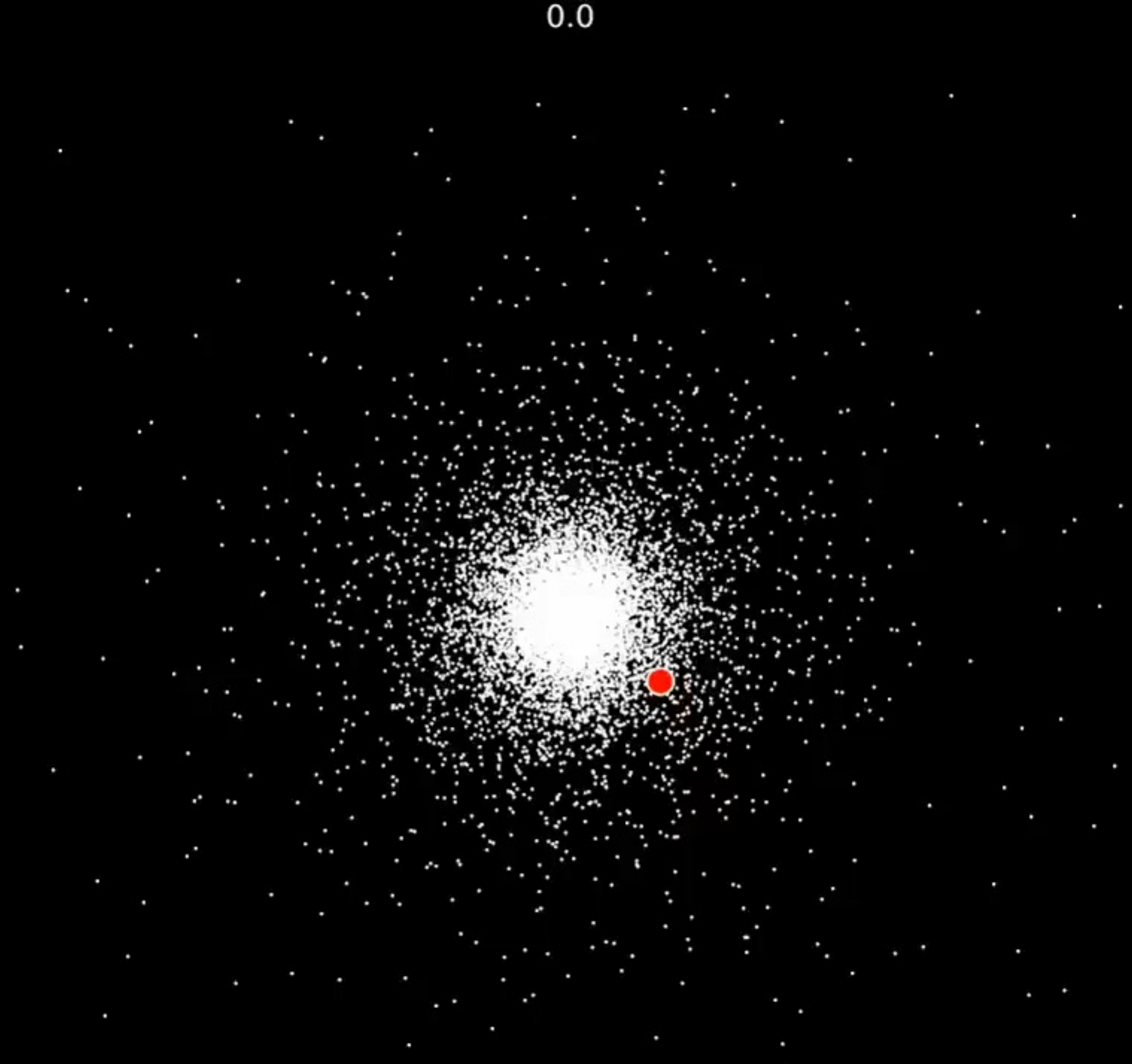
Kocsis & Tremaine (2011)
Szolgyen & Kocsis (2018)
Meiron & Kocsis (2019)

Flattened systems



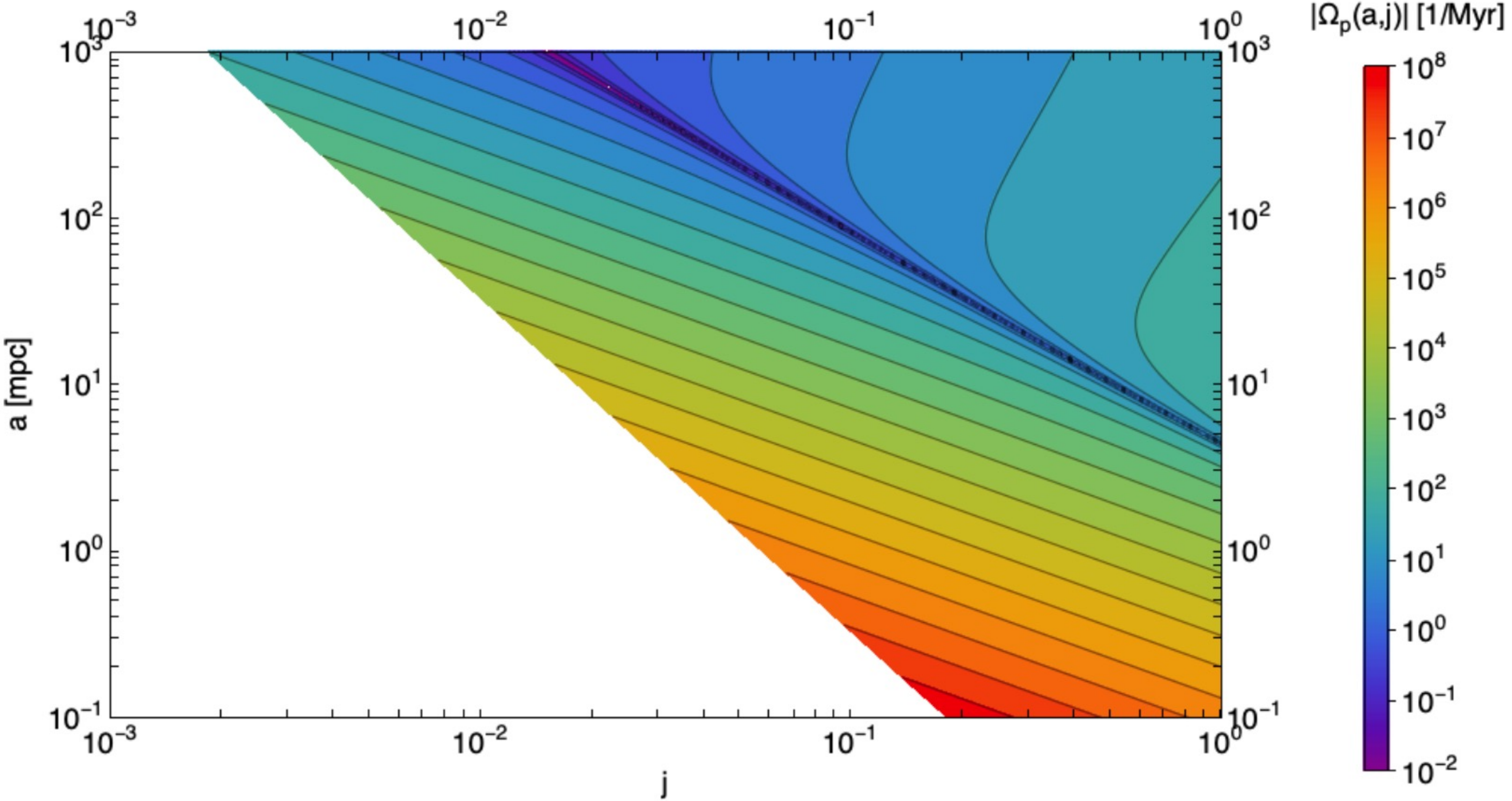


Dejonghe & de Zeeuw (1988)
Sanders & Binney (2016)
Vasiliev (2019)

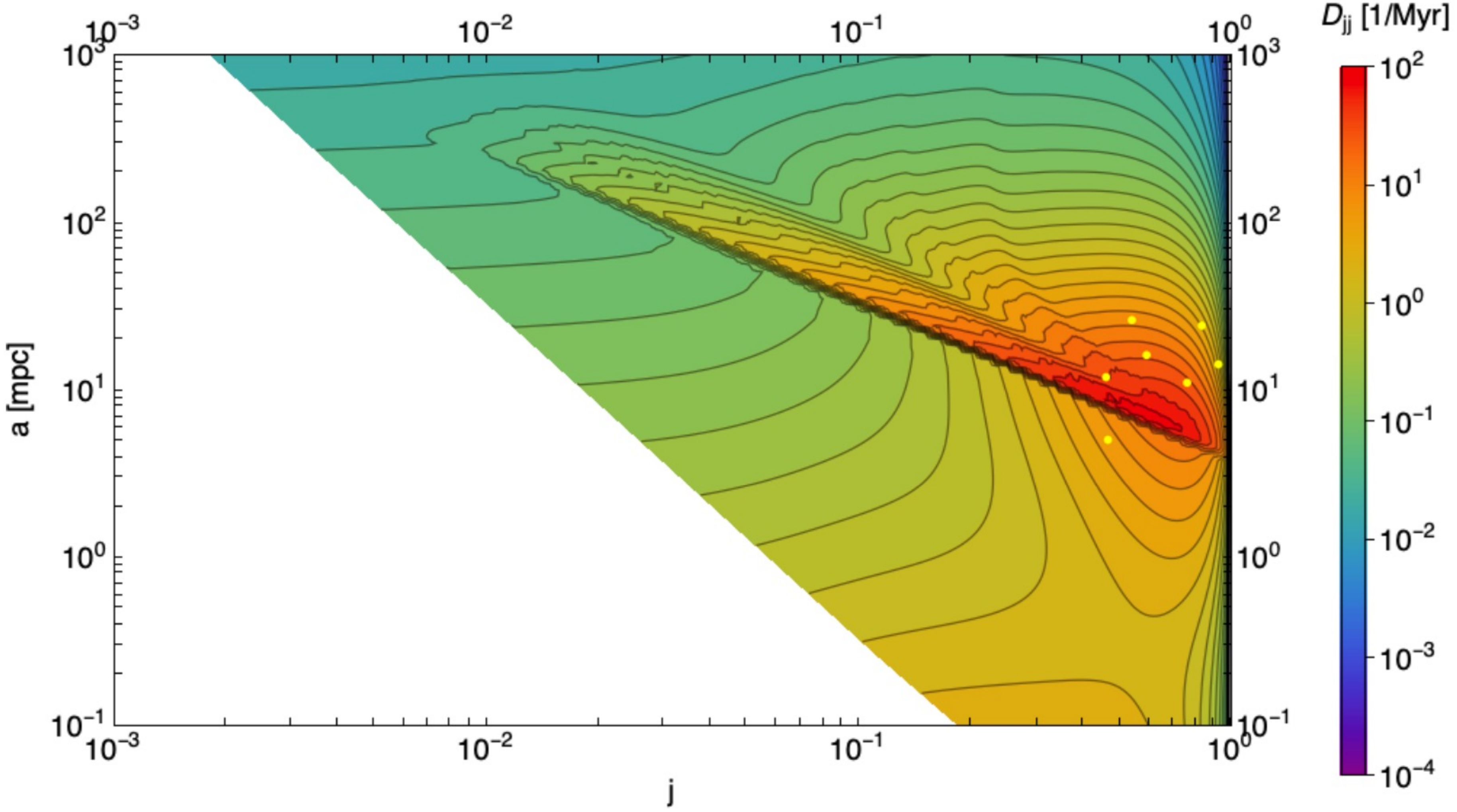


Thank you !

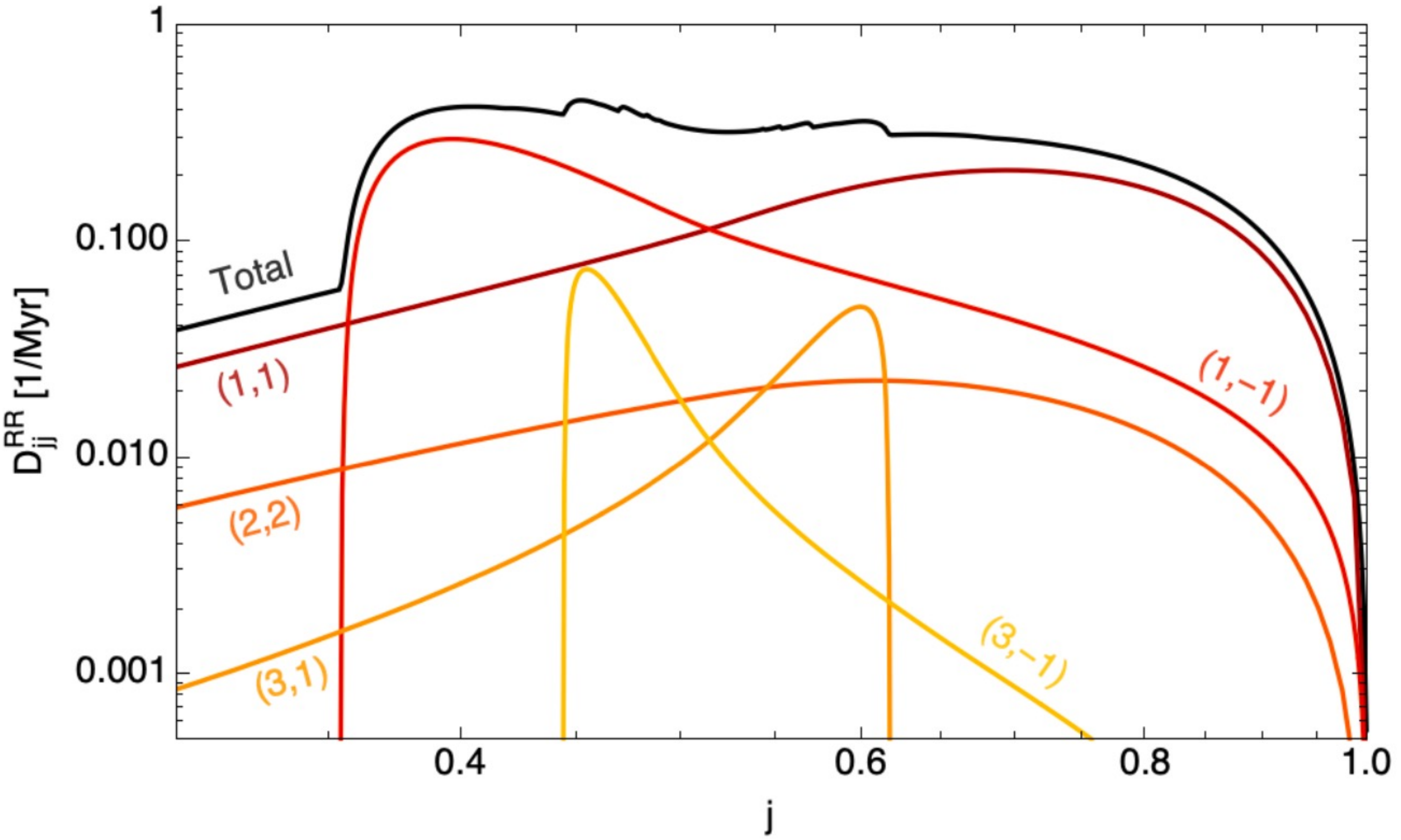
Galactic nucleus: precession frequency



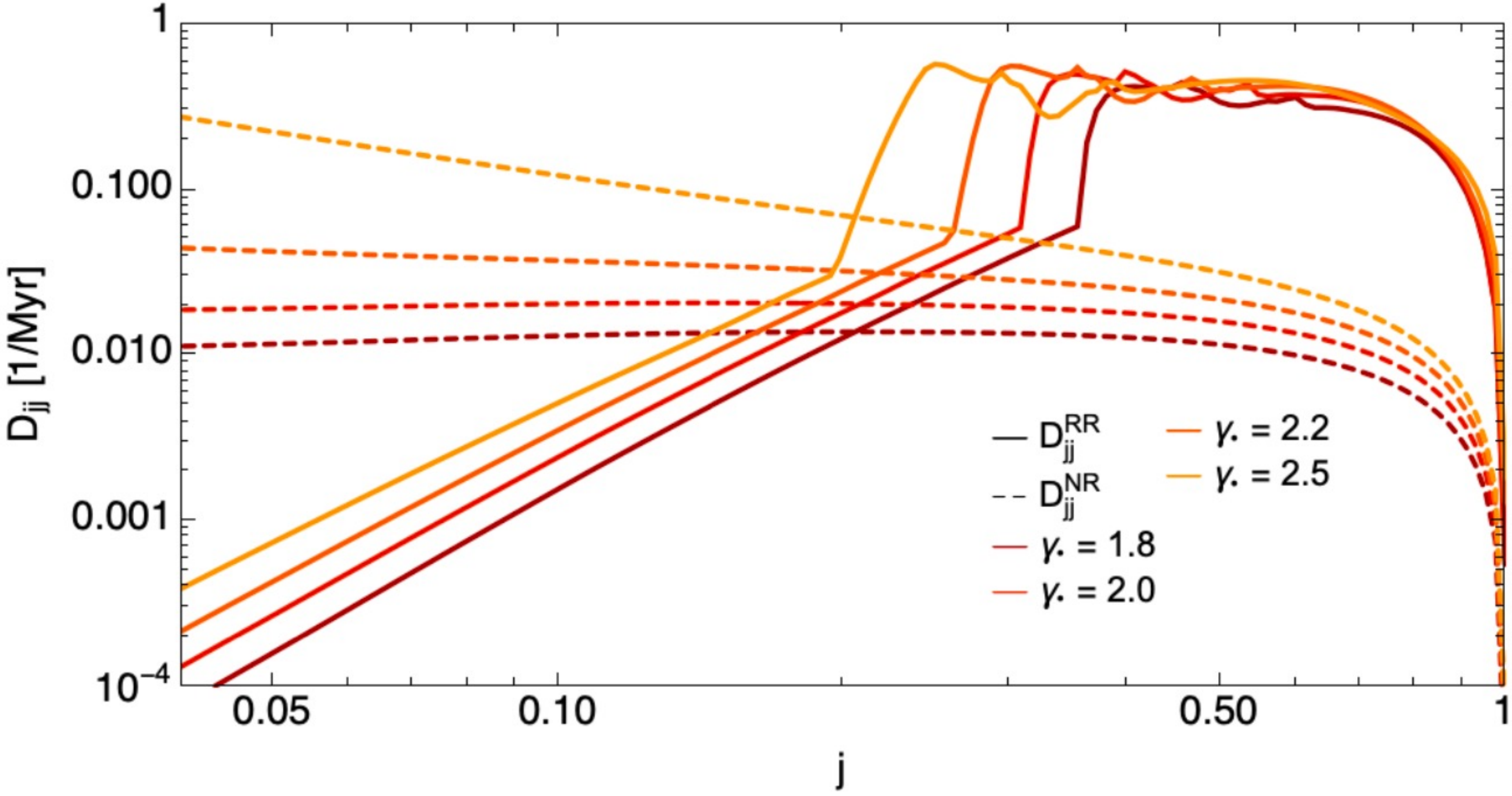
Galactic nucleus: diffusion coefficient



Galactic nucleus: diffusion coefficient

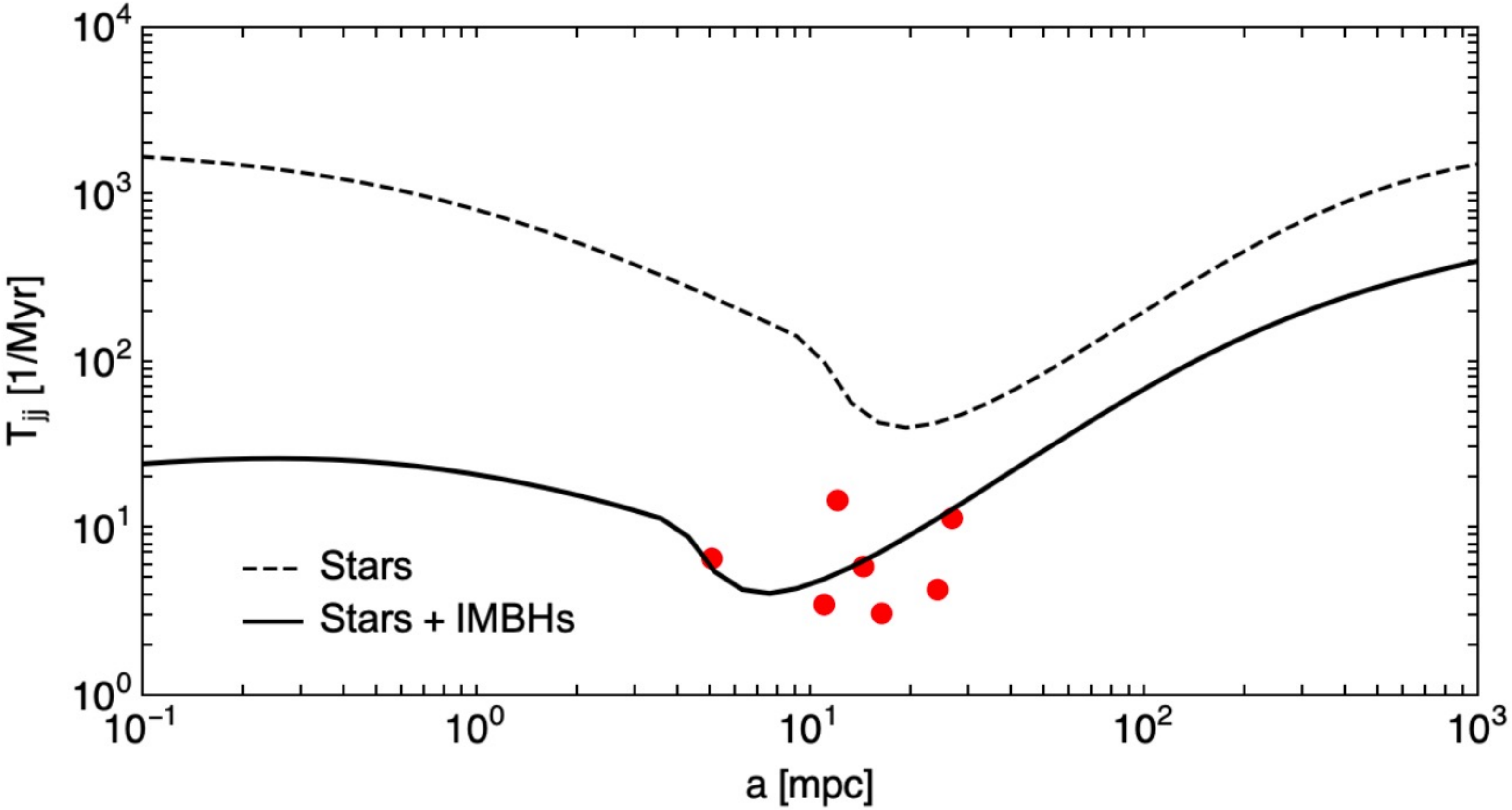


Galactic nucleus: diffusion coefficient



Galactic nucleus: diffusion time

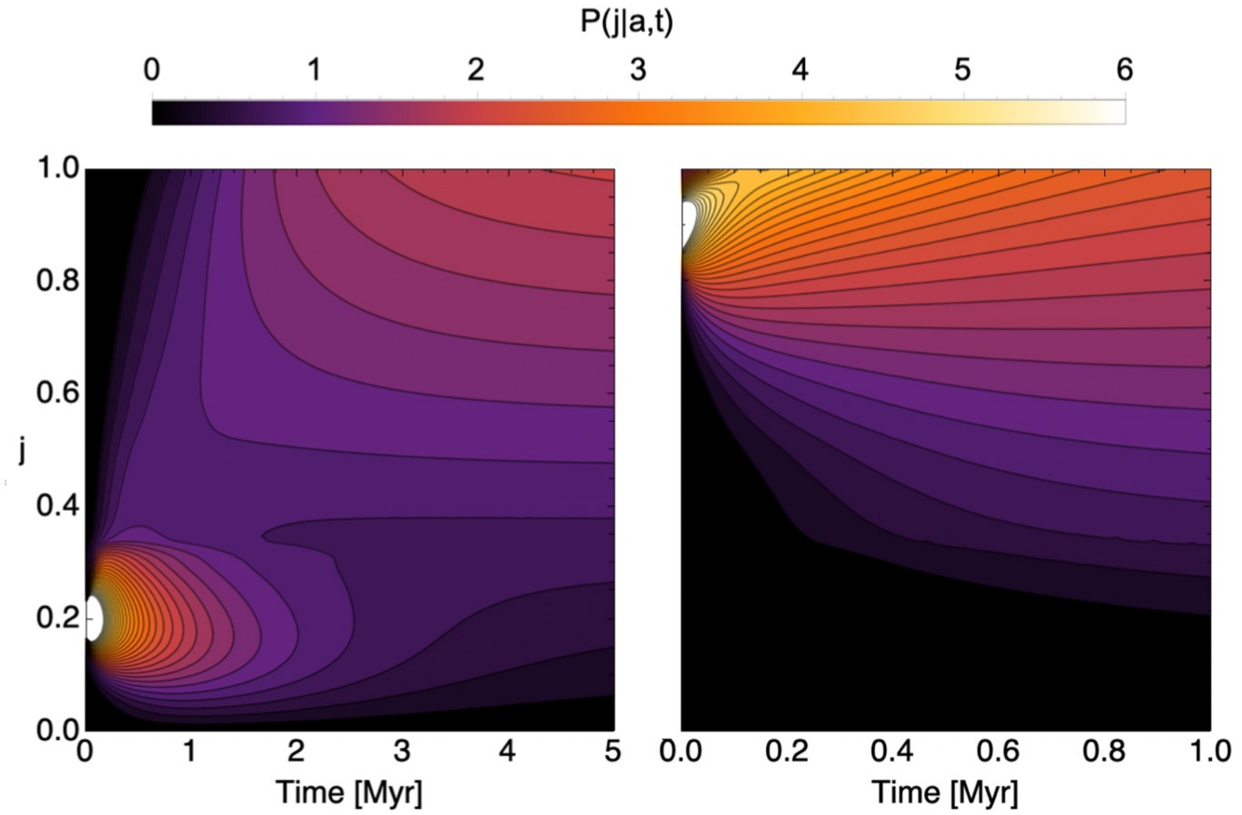
$$T_{jj}(a) = \frac{1}{D_{jj}^{\mathrm{iso}}(a)} \quad ; \quad D_{jj}^{\mathrm{iso}}(a) = \int_0^1 \mathrm{d}j\, f(j;a) D_{jj}(a,j),$$

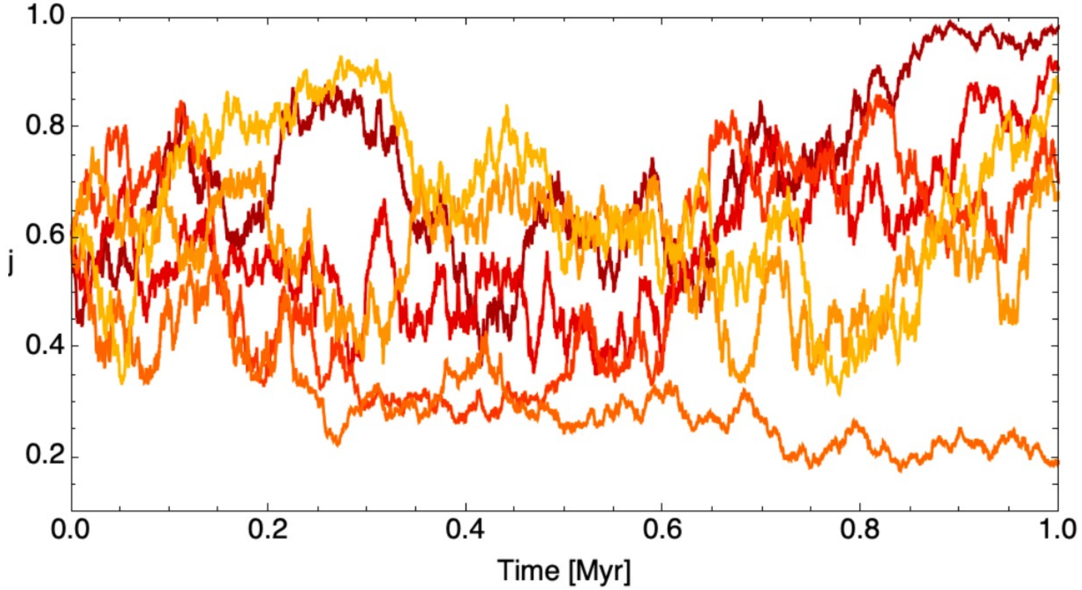


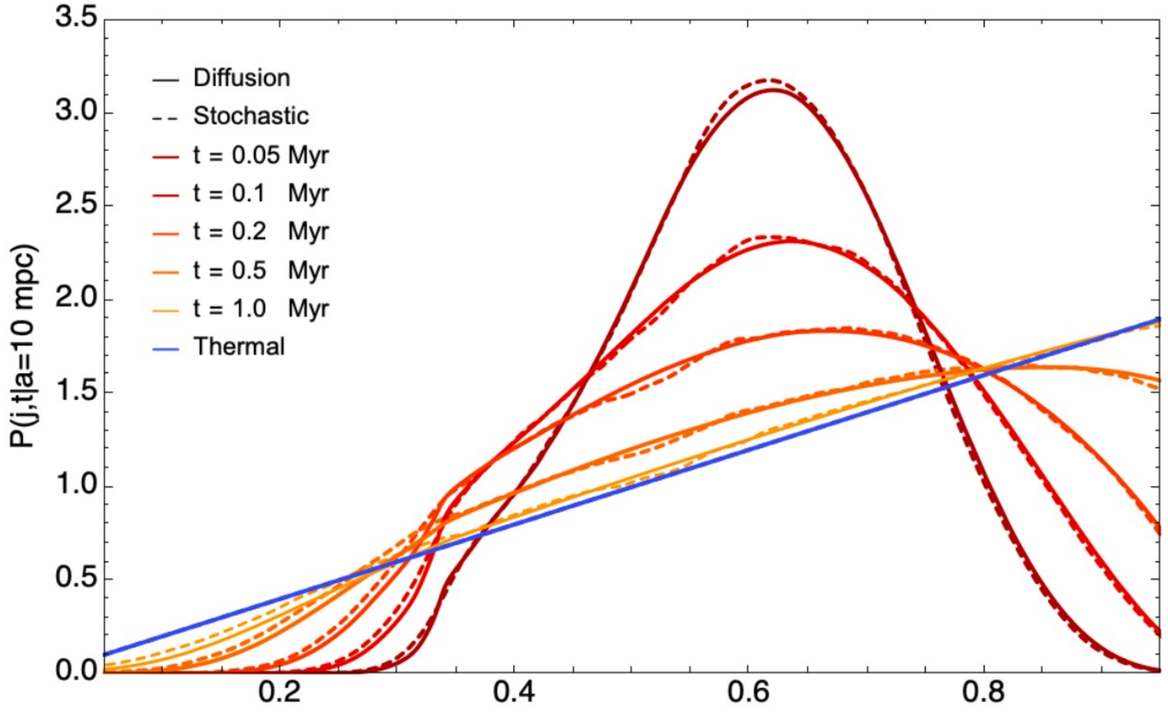
Galactic nucleus: DF

$$\frac{\partial P(j,t\,|\,a)}{\partial t} = \frac{1}{2}\frac{\partial}{\partial j}\left[j\,D_{jj}(a,j)\,\frac{\partial}{\partial j}\left(\frac{P(j,t\,|\,a)}{j}\right)\right]$$

$$D_{jj}(a,j) = D_{jj}^{\mathrm{RR}}(a,j) + D_{jj}^{\mathrm{NR}}(a,j)$$





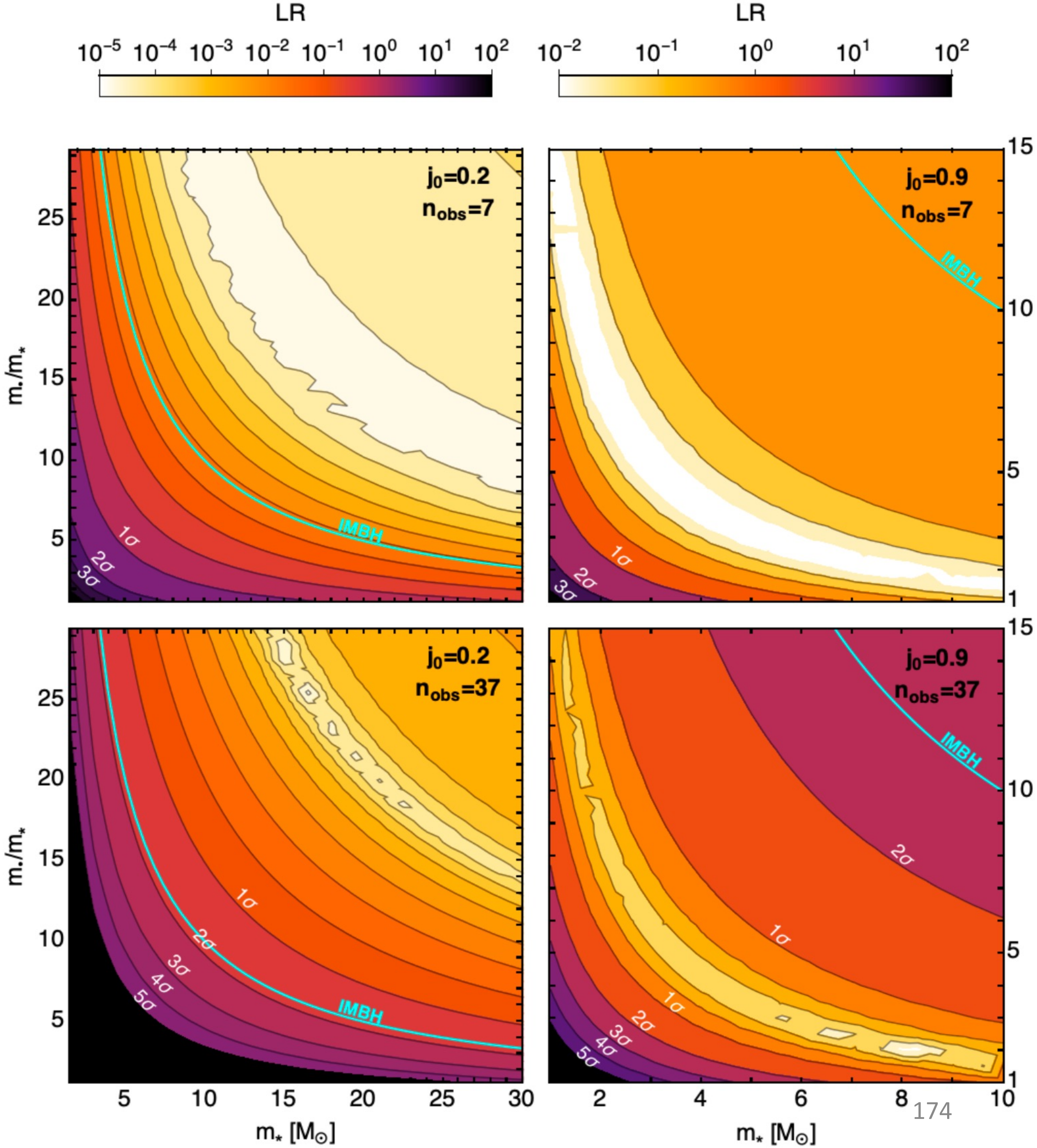


Galactic nucleus: LR

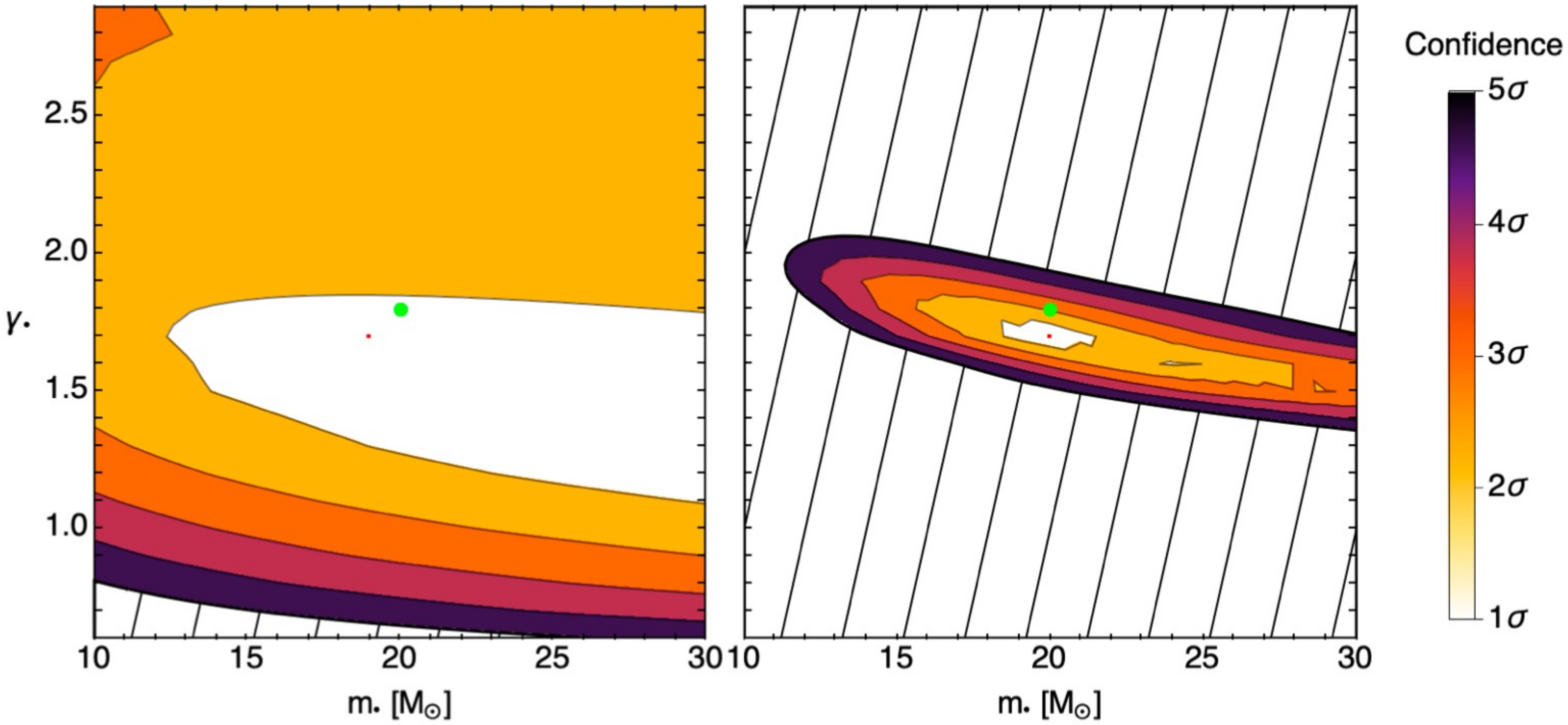
$$M_i(< a) = M_i(< a_0)(a/a_0)^{3-\gamma_i}$$

$$L(\boldsymbol{\alpha}) = \prod_k P(j_k, T_k \,|\, a_k)$$

$$\lambda_\mathrm{R}(\boldsymbol{\alpha}) = 2\ln\left(L_\mathrm{max}/L[\boldsymbol{\alpha}]\right)$$



Galactic nucleus: data convergence



GC: local velocity deflection

$$\langle \Delta v_{\parallel} \rangle = -8\pi m G^2 \ln\Lambda \int_0^{\pi} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\phi \int_0^{w_{\max}} \mathrm{d}w \, \sin\varphi \cos\varphi \, F_{\mathrm{tot}}(r, E', L'),$$

$$\langle (\Delta v_{\parallel})^2 \rangle = 4\pi m G^2 \ln\Lambda \int_0^{\pi} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\phi \int_0^{w_{\max}} \mathrm{d}w \, w \sin^3\varphi F_{\mathrm{tot}}(r, E', L'),$$

$$\langle (\Delta v_{\perp})^2 \rangle = 4\pi m G^2 \ln\Lambda \int_0^{\pi} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\phi \int_0^{w_{\max}} \mathrm{d}w \, w \sin\varphi (1 + \cos^2\varphi) F_{\mathrm{tot}}(r, E', L'),$$

$$w_{\max} = v\cos\varphi + \sqrt{v^2\cos^2\varphi - 2E}$$

$$E'(r, \boldsymbol{v}, \boldsymbol{v}') = \psi(r) + \frac{v^2}{2} + \frac{w^2}{2} - vw\cos\varphi,$$

$$L'(r, \boldsymbol{v}, \boldsymbol{v}') = r\sqrt{(w\sin\varphi\cos\phi)^2 + \left(v_{\mathrm{t}} + \frac{v_r}{v} w \sin\varphi \sin\phi - \frac{v_{\mathrm{t}}}{v} w \cos\varphi\right)^2}.$$

GC: local invariant diffusion

$$\begin{aligned}
\langle \Delta E \rangle &= \frac{1}{2}\langle (\Delta v_{\|})^2 \rangle + \frac{1}{2}\langle (\Delta v_{\perp})^2 \rangle + v \langle \Delta v_{\|} \rangle, \\
\langle (\Delta E)^2 \rangle &= v^2 \langle (\Delta v_{\|})^2 \rangle, \\
\langle \Delta L \rangle &= \frac{L}{v}\langle \Delta v_{\|} \rangle + \frac{r^2}{4L}\langle (\Delta v_{\perp})^2 \rangle, \\
\langle (\Delta L^2 \rangle &= \frac{L^2}{v^2}\langle (\Delta v_{\|})^2 \rangle + \frac{1}{2}\frac{r^2 v_r^2}{v^2}\langle (\Delta v_{\perp})^2 \rangle, \\
\langle \Delta E \Delta L \rangle &= L \langle (\Delta v_{\|})^2 \rangle.
\end{aligned}$$

GC: local invariant diffusion (rotation)

$$\begin{aligned}
\langle \Delta E \rangle &= \frac{1}{2}\langle (\Delta v_{\|})^2 \rangle + \frac{1}{2}\langle (\Delta v_{\perp})^2 \rangle + v\langle \Delta v_{\|} \rangle, \\
\langle (\Delta E)^2 \rangle &= v^2 \langle (\Delta v_{\|})^2 \rangle, \\
\langle \Delta L \rangle &= \frac{L}{v}\langle \Delta v_{\|} \rangle + \frac{r^2}{4L}\langle (\Delta v_{\perp})^2 \rangle, \\
\langle (\Delta L^2 \rangle &= \frac{L^2}{v^2}\langle (\Delta v_{\|})^2 \rangle + \frac{1}{2}\frac{r^2 v_r^2}{v^2}\langle (\Delta v_{\perp})^2 \rangle, \\
\langle \Delta E \Delta L \rangle &= L \langle (\Delta v_{\|})^2 \rangle .
\end{aligned}$$

$$\begin{aligned}
\langle \Delta L_z \rangle &= \frac{L_z}{v}\langle \Delta v_{\|} \rangle, \\
\langle (\Delta L_z^2 \rangle &= \left(\frac{L_z}{L}\right)^2 \left[\frac{L^2}{v^2}\langle (\Delta v_{\|})^2 \rangle + \frac{1}{2}\frac{r^2 v_r^2}{v^2}\langle (\Delta v_{\perp})^2 \rangle\right] + \frac{r^2 \sin^2\theta}{2}\left(1 - \frac{L_z^2}{L^2}\right)\langle (\Delta v_{\perp})^2 \rangle \\
\langle \Delta E \Delta L_z \rangle &= L_z \langle (\Delta v_{\|})^2 \rangle, \\
\langle \Delta L \Delta L_z \rangle &= \frac{L_z}{L}\left(\frac{L^2}{v^2}\langle (\Delta v_{\|})^2 \rangle + \frac{1}{2}\frac{r^2 v_r^2}{v^2}\langle (\Delta v_{\perp})^2 \rangle\right).
\end{aligned}$$

GC: orbit-average

$$D_X(\boldsymbol{J}) = \frac{\Omega_r}{\pi} \int_{r_\mathrm{p}}^{r_\mathrm{a}} \frac{\mathrm{d}r}{|v_r|} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \langle \Delta X \rangle (r, \theta, \boldsymbol{J})$$

$$\begin{aligned}
D_{J_\mathrm{r}} &= \frac{\partial J_\mathrm{r}}{\partial E} D_E + \frac{\partial J_\mathrm{r}}{\partial L} D_L + \frac{1}{2} \frac{\partial^2 J_\mathrm{r}}{\partial E^2} D_{EE} + \frac{1}{2} \frac{\partial^2 J_\mathrm{r}}{\partial L^2} D_{LL} + \frac{\partial^2 J_\mathrm{r}}{\partial E \partial L} D_{EL}, \\
D_{J_\mathrm{r} L} &= \frac{\partial J_\mathrm{r}}{\partial E} D_{EL} + \frac{\partial J_\mathrm{r}}{\partial L} D_{LL}, \\
D_{J_\mathrm{r} J_\mathrm{r}} &= \left(\frac{\partial J_\mathrm{r}}{\partial E} \right)^2 D_{EE} + 2 \frac{\partial J_\mathrm{r}}{\partial E} \frac{\partial J_\mathrm{r}}{\partial L} D_{EL} + \left(\frac{\partial J_\mathrm{r}}{\partial L} \right)^2 D_{LL}.
\end{aligned}$$

GC: Chandrasekhar theory

$$\frac{\partial F(\boldsymbol{J})}{\partial t} = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{J}) = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \left[\boldsymbol{D}_1(\boldsymbol{J})\, F(\boldsymbol{J}) - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{J}} \cdot \left(\boldsymbol{D}_2(\boldsymbol{J})\, F(\boldsymbol{J})\right)\right],$$

$$\boldsymbol{D}_1(\boldsymbol{J}) = \begin{pmatrix} D_{J_\mathrm{r}} \\ D_L \end{pmatrix}, \quad \boldsymbol{D}_2(\boldsymbol{J}) = \begin{pmatrix} D_{J_\mathrm{r} J_\mathrm{r}} & D_{J_\mathrm{r} L} \\ D_{J_\mathrm{r} L} & D_{LL} \end{pmatrix}$$

GC: Chandrasekhar theory (rotation)

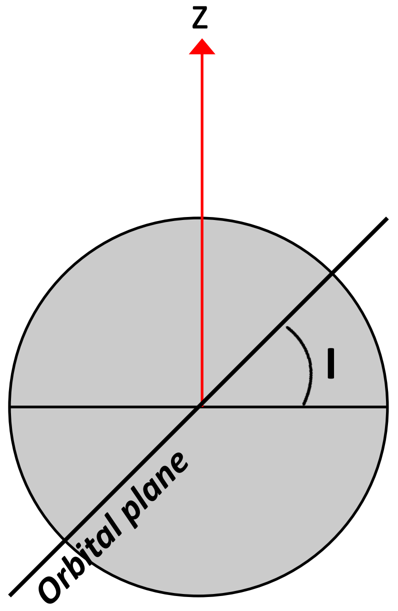
$$\frac{\partial F(\boldsymbol{J})}{\partial t} = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{J}) = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \left[\boldsymbol{D}_1(\boldsymbol{J})\, F(\boldsymbol{J}) - \frac{1}{2} \frac{\partial}{\partial \boldsymbol{J}} \cdot \Big(\boldsymbol{D}_2(\boldsymbol{J})\, F(\boldsymbol{J}) \Big) \right],$$

$$\boldsymbol{D}_1(\boldsymbol{J}) = \begin{pmatrix} D_{J_\mathrm{r}} \\ D_L \end{pmatrix}, \quad \boldsymbol{D}_2(\boldsymbol{J}) = \begin{pmatrix} D_{J_\mathrm{r} J_\mathrm{r}} & D_{J_\mathrm{r} L} \\ D_{J_\mathrm{r} L} & D_{LL} \end{pmatrix}$$

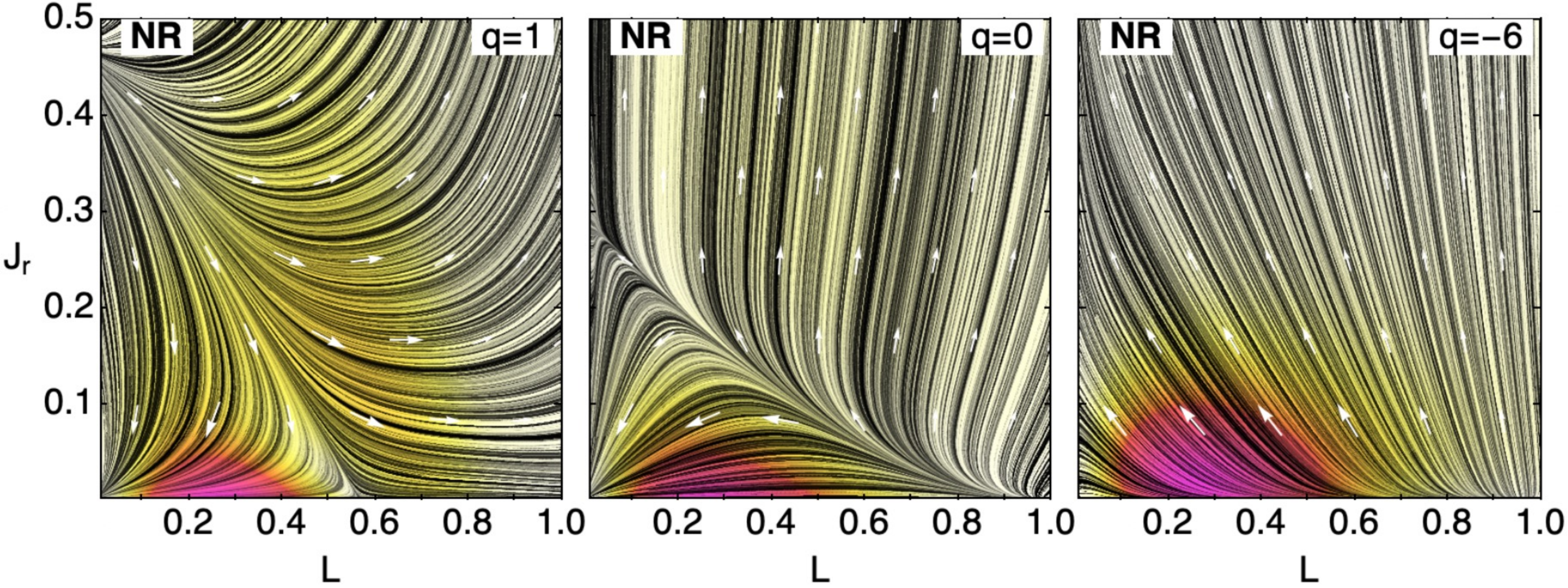
3D

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{J}) = -\frac{\partial}{\partial \boldsymbol{J}} \cdot \left[\boldsymbol{D}_1(\boldsymbol{J})\, F - \frac{1}{2} \frac{\partial}{\partial \boldsymbol{J}} \cdot \Big(\boldsymbol{D}_2(\boldsymbol{J})\, F \Big) \right]$$

$$\boldsymbol{D}_1(\boldsymbol{J}) = \begin{pmatrix} D_{J_\mathrm{r}} \\ D_L \\ D_{\cos I} \end{pmatrix}, \quad \boldsymbol{D}_2(\boldsymbol{J}) = \begin{pmatrix} D_{J_\mathrm{r} J_\mathrm{r}} & D_{J_\mathrm{r} L} & 0 \\ D_{J_\mathrm{r} L} & D_{LL} & 0 \\ 0 & 0 & D_{\cos I \cos I} \end{pmatrix},$$

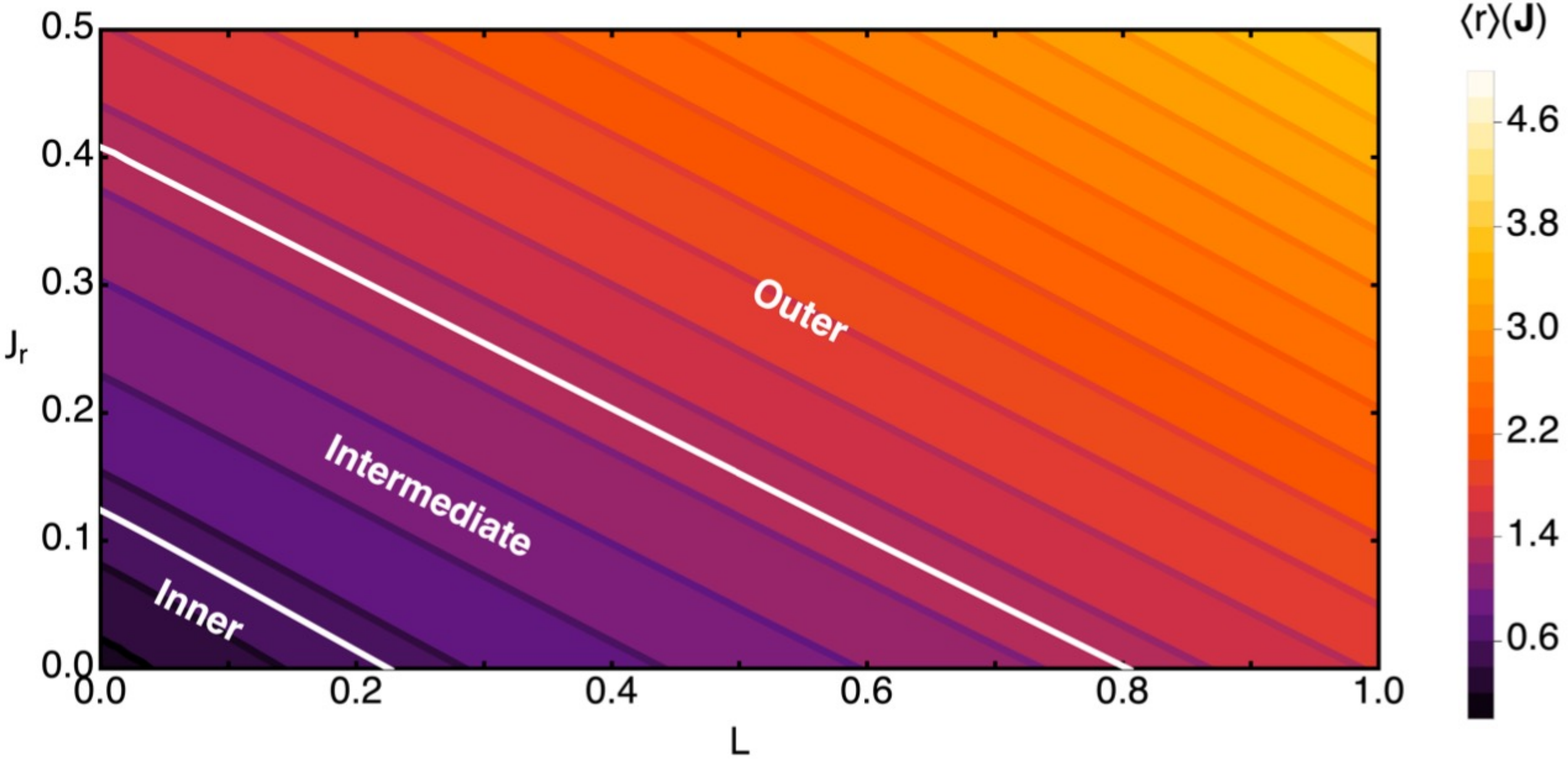


GC: flux

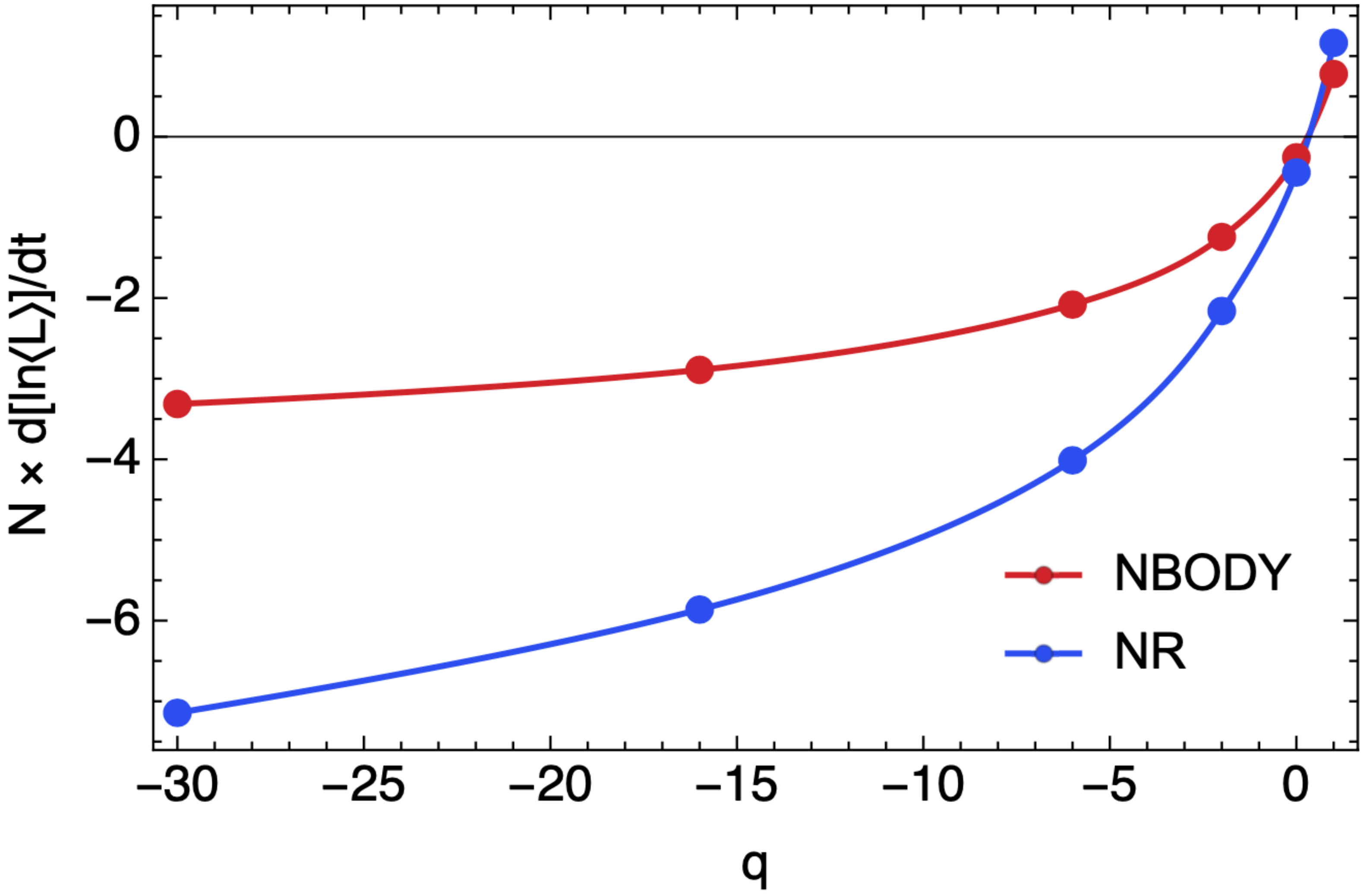


NR
q=1
NR
q=0
NR
q=−6
0.5
0.4
0.3
0.2
0.1
J_r
0.2 0.4 0.6 0.8 1.0
L
0.2 0.4 0.6 0.8 1.0
L
0.2 0.4 0.6 0.8 1.0
L

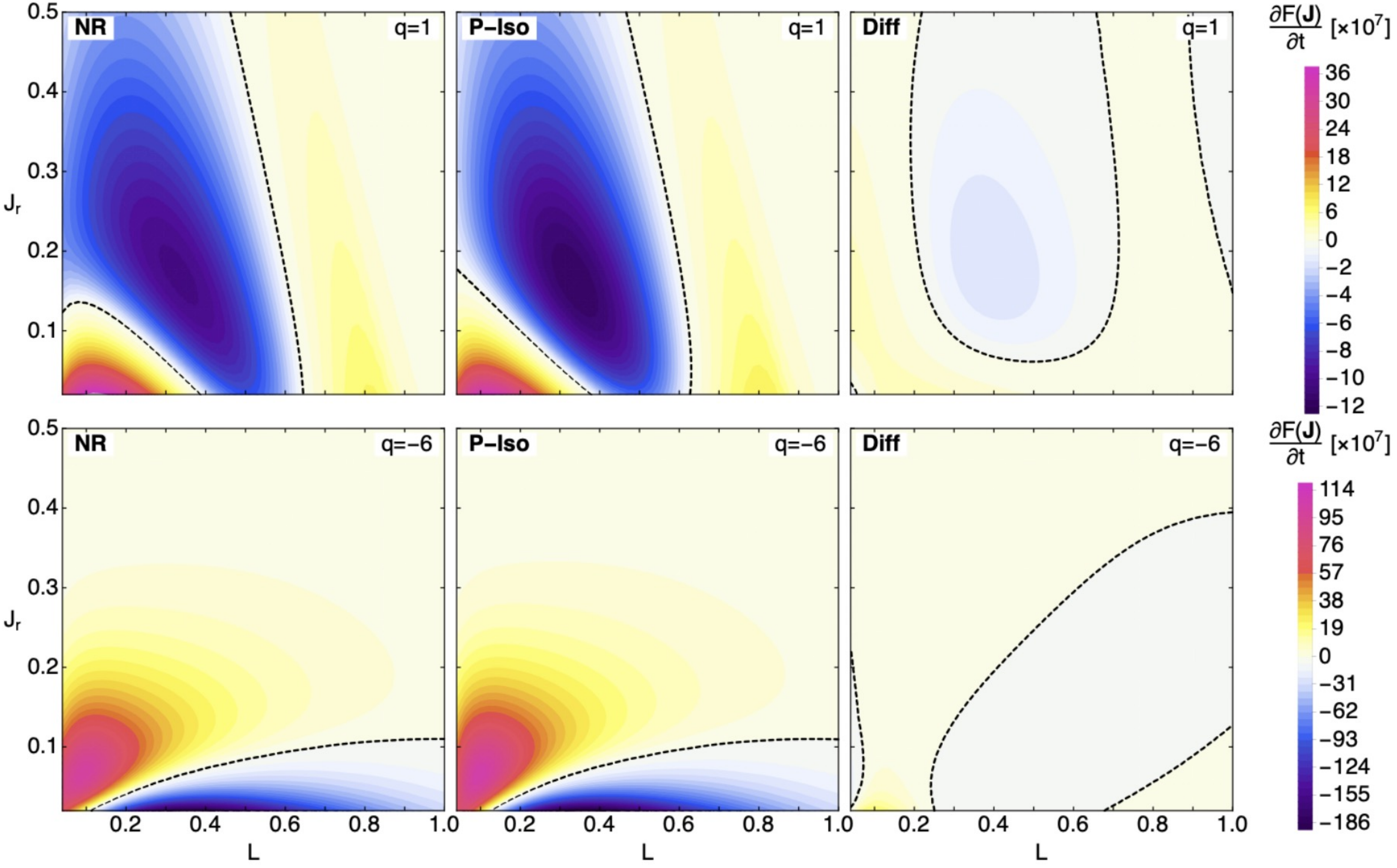
GC: cluster regions



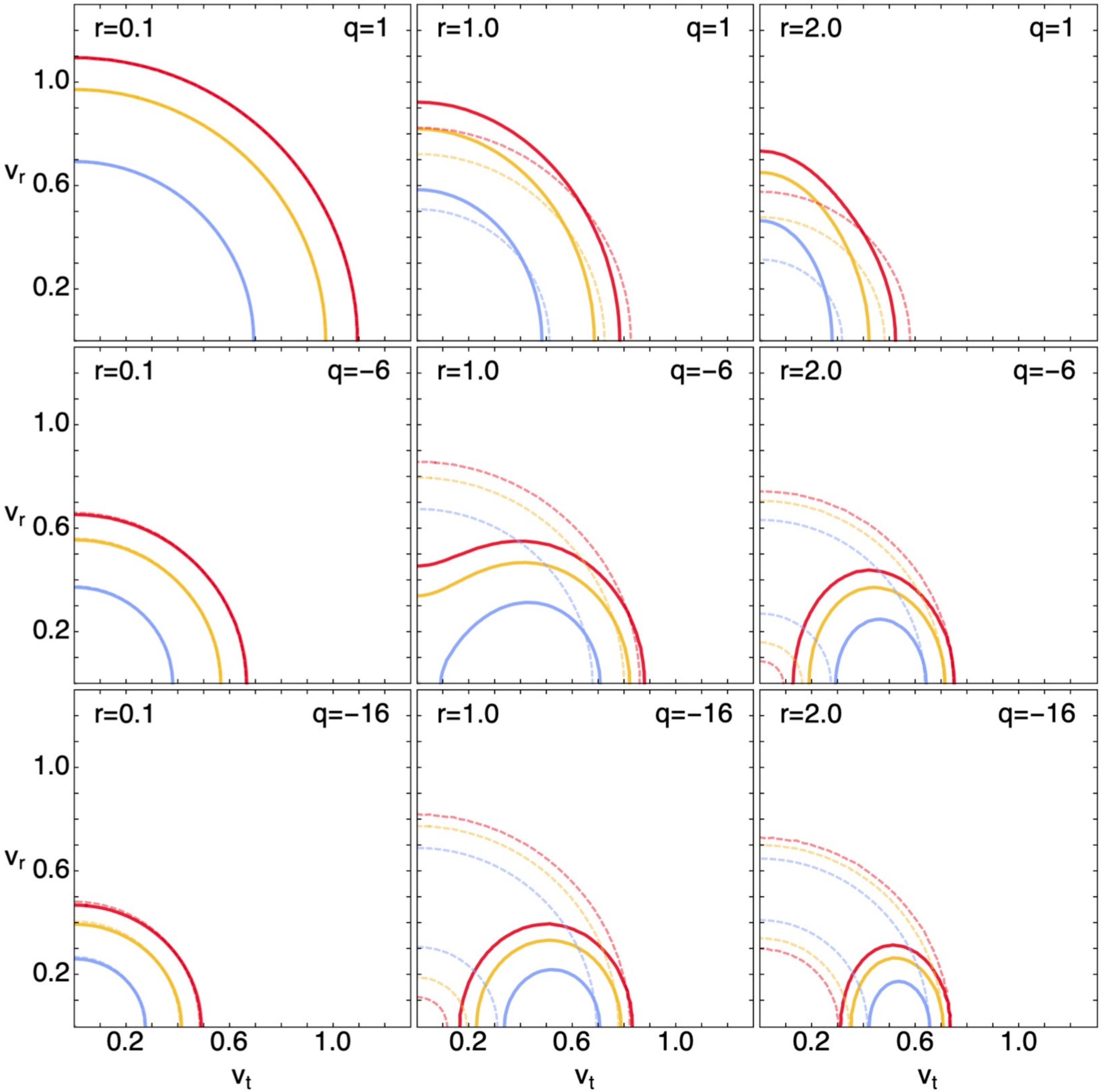
GC: isotropisation



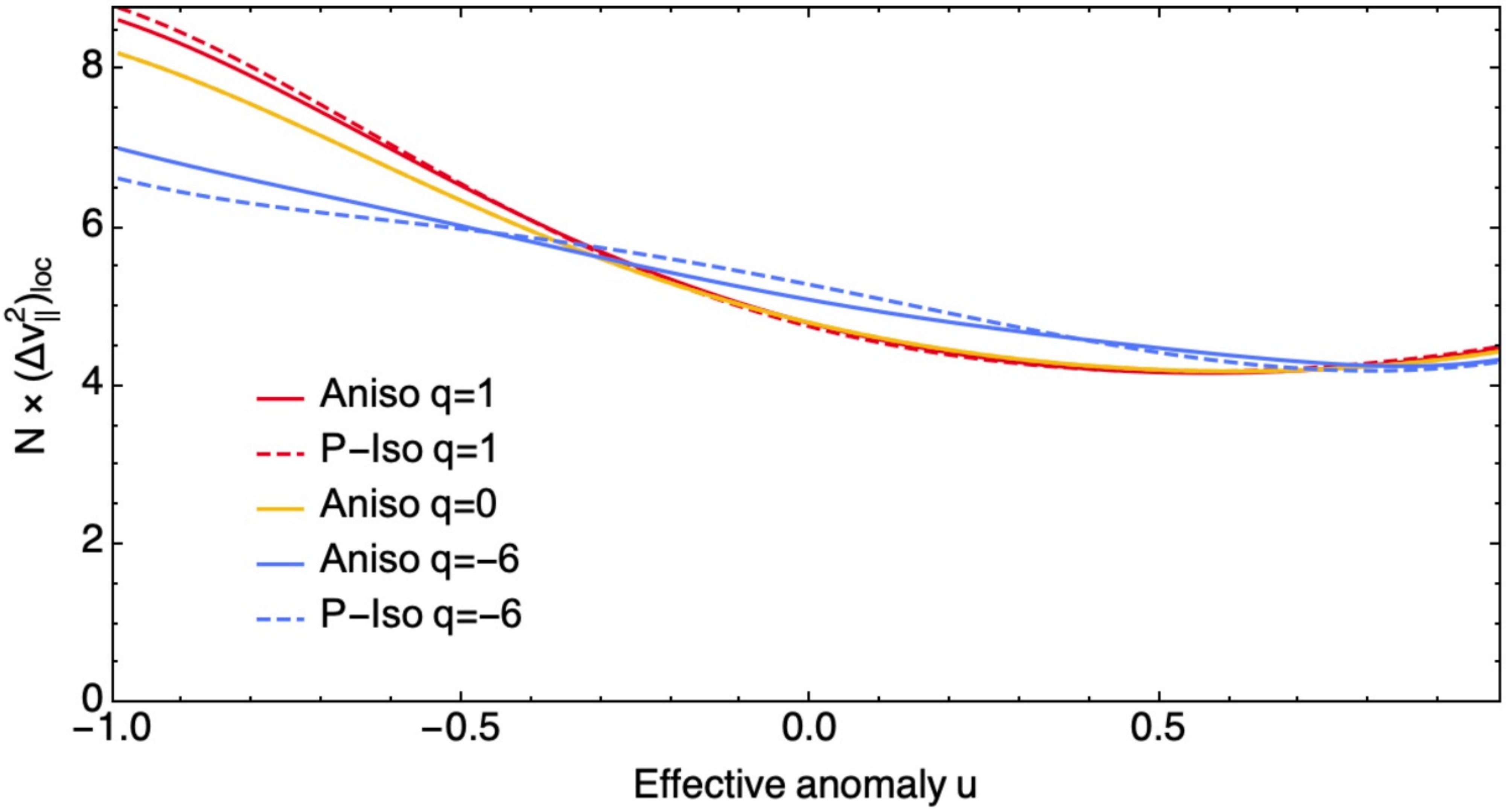
GC: pseudo-isotropic method



GC: P-ISO DF

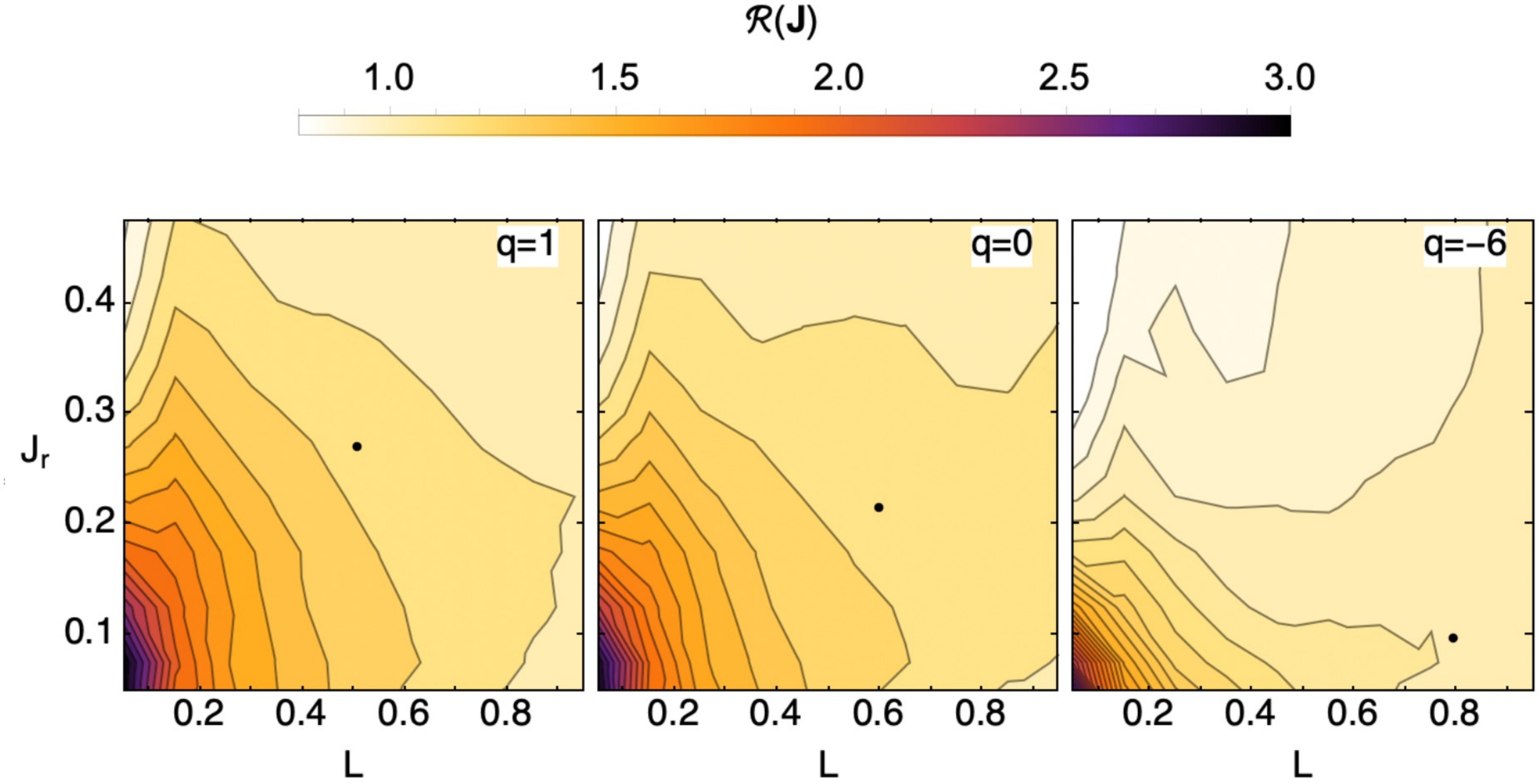

r=0.1
q=1
r=1.0
q=1
r=2.0
q=1
r=0.1
q=−6
r=1.0
q=−6
r=2.0
q=−6
r=0.1
q=−16
r=1.0
q=−16
r=2.0
q=−16
v_r
1.0
0.6
0.2
v_t
0.2
0.6
1.0

GC: P-Iso local deflections



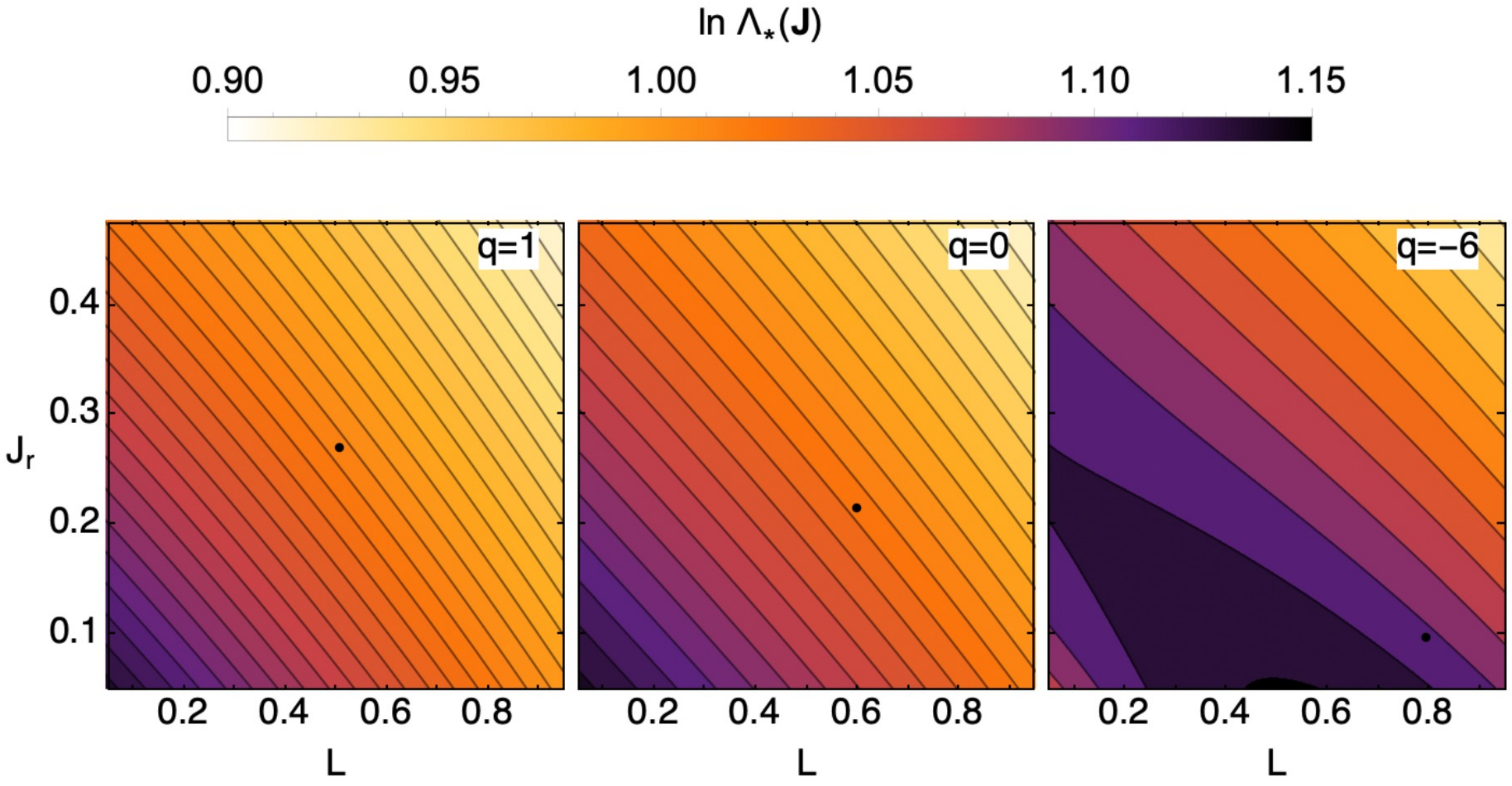
GC: P-Iso local deflections

$$\mathcal{R}(\boldsymbol{J}) \sim \frac{\ell\, D^{\ell}_{\mathrm{RR}}(\boldsymbol{J})}{D_{\mathrm{NR}}(\boldsymbol{J})/\ln\Lambda}$$



GC: P-Iso local deflections

$$\ln \Lambda_\star(\boldsymbol{J}) = \ln \frac{b\,\langle\sigma^2\rangle(\boldsymbol{J})}{2Gm}.$$



Bars: Euler-Poisson equations

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{v}) = 0\,,$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\Sigma}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi$$

$$\Delta\Phi_{\rm disc} = 4\pi G \Sigma \delta_{\rm D}(z),$$

Bars: Linear theory

$$X(r,\theta,t) = X_0(r) + \delta X(r,\theta,t) \quad ; \quad \delta X(r,\theta,t) = \sum_{m\in\mathbb{Z}} X_m(r)\,\mathrm{e}^{\mathrm{i}(m\theta-\omega_m t)}$$

$$\delta\Sigma_m^{\mathrm{disc}}(r) = \frac{M(1-p)}{2\pi a_\mathrm{d}^2}\left(\frac{1-\xi}{2}\right)^{3/2}\sum_{n=|m|}^{\infty} a_n^m P_n^{|m|}(\xi),$$

$$\int_{-1}^{1}\mathrm{d}\xi\, P_n^{|m|}(\xi)P_l^{|m|}(\xi) = \delta_{nl}$$

$$\delta\Phi_m^{\mathrm{disc}}(r) = -\frac{GM(1-p)}{a_\mathrm{d}}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\frac{a_n^m}{2n+1}P_n^{|m|}(\xi),$$

$$\delta\psi_m(r) = \frac{4\alpha}{3}\left(\frac{M}{2\pi a_\mathrm{d}^2}\right)^{1/3}(1-p)^{1/3}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty} a_n^m P_n^{|m|}(\xi),$$

$$\delta\Psi_m(r) = \frac{GM(1-p)}{a_\mathrm{d}}\left(\frac{1-\xi}{2}\right)^{1/2}\sum_{n=|m|}^{\infty}\left[\frac{\varepsilon_0}{3(1-p)^{2/3}} - \frac{1}{2n+1}\right]a_n^m P_n^{|m|}(\xi).$$

$$\delta v_{r,m}(r) = \mathrm{i}\frac{m}{|m|}\left(\frac{GM(1-p)}{a_\mathrm{d}}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} b_n^m P_n^{|m|}(\xi),$$

$$\delta v_{\mathrm{t},m}(r) = \left(\frac{GM(1-p)}{a_\mathrm{d}}\right)^{1/2}\left(\frac{1+\xi}{2}\right)^{-1/2}\left(\frac{1-\xi}{2}\right)^{1/4}\sum_{n=|m|}^{\infty} c_n^m P_n^{|m|}(\xi).$$

Bars: Linear theory

$$\mathrm{i}(-\omega_m + m\Omega)\delta\Sigma_m^{\mathrm{disc}} + \frac{1}{r}\frac{\mathrm{d}(r\Sigma^0\delta v_{r,m})}{\mathrm{d}r} + \frac{\mathrm{i}m\Sigma^0\delta v_{\mathrm{t},m}}{r} = 0,$$

$$\frac{\mathrm{d}\delta\Psi_m}{\mathrm{d}r} + \mathrm{i}(-\omega_m + m\Omega)\delta v_{r,m} - 2\Omega\delta v_{\mathrm{t},m} = 0,$$

$$\mathrm{i}m\frac{\delta\Psi_m}{r} + \frac{\kappa^2}{2\Omega}\delta v_{r,m} + \mathrm{i}(-\omega_m + m\Omega)\delta v_{\mathrm{t},m} = 0,$$

$$\sum_{n=|m|}^{\infty} A_{ln}a_n^m + \sum_{n=|m|}^{\infty} B_{ln}b_n^m + \sum_{n=|m|}^{\infty} C_{ln}c_n^m = \widehat{\omega}a_l^m,$$

$$\sum_{n=|m|}^{\infty} D_{ln}a_n^m + \sum_{n=|m|}^{\infty} A_{ln}b_n^m + \sum_{n=|m|}^{\infty} F_{ln}c_n^m = \widehat{\omega}b_l^m,$$

$$\sum_{n=|m|}^{\infty} G_{ln}a_n^m + \sum_{n=|m|}^{\infty} H_{ln}b_n^m + \sum_{n=|m|}^{\infty} A_{ln}c_n^m = \widehat{\omega}c_l^m.$$

$$\mathbf{M}\,\mathbf{a} = \widehat{\omega}\,\mathbf{a}$$

Bars: matrix coefficients

$$A_{ln} = |m| \int_{-1}^{1} \mathrm{d}\xi P_l^{|m|}(\xi) \widehat{\Omega}(\xi) P_n^{|m|}(\xi),$$

$$B_{ln} = 4\sqrt{1-p} \int_{-1}^{1} \mathrm{d}\xi P_l^{|m|}(\xi) \left(\frac{1-\xi}{2}\right)^{1/2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\frac{1-\xi}{2}\right)^{5/4} P_n^{|m|}(\xi) \right],$$

$$C_{ln} = |m|\sqrt{1-p} \int_{-1}^{1} \mathrm{d}\xi P_l^{|m|}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} \left(\frac{1+\xi}{2}\right)^{-1} P_n^{|m|}(\xi),$$

$$D_{ln} = 4\sqrt{1-p} \left(\frac{1}{2n+1} - \frac{\varepsilon_0}{3} \frac{1}{(1-p)^{2/3}} \right)$$

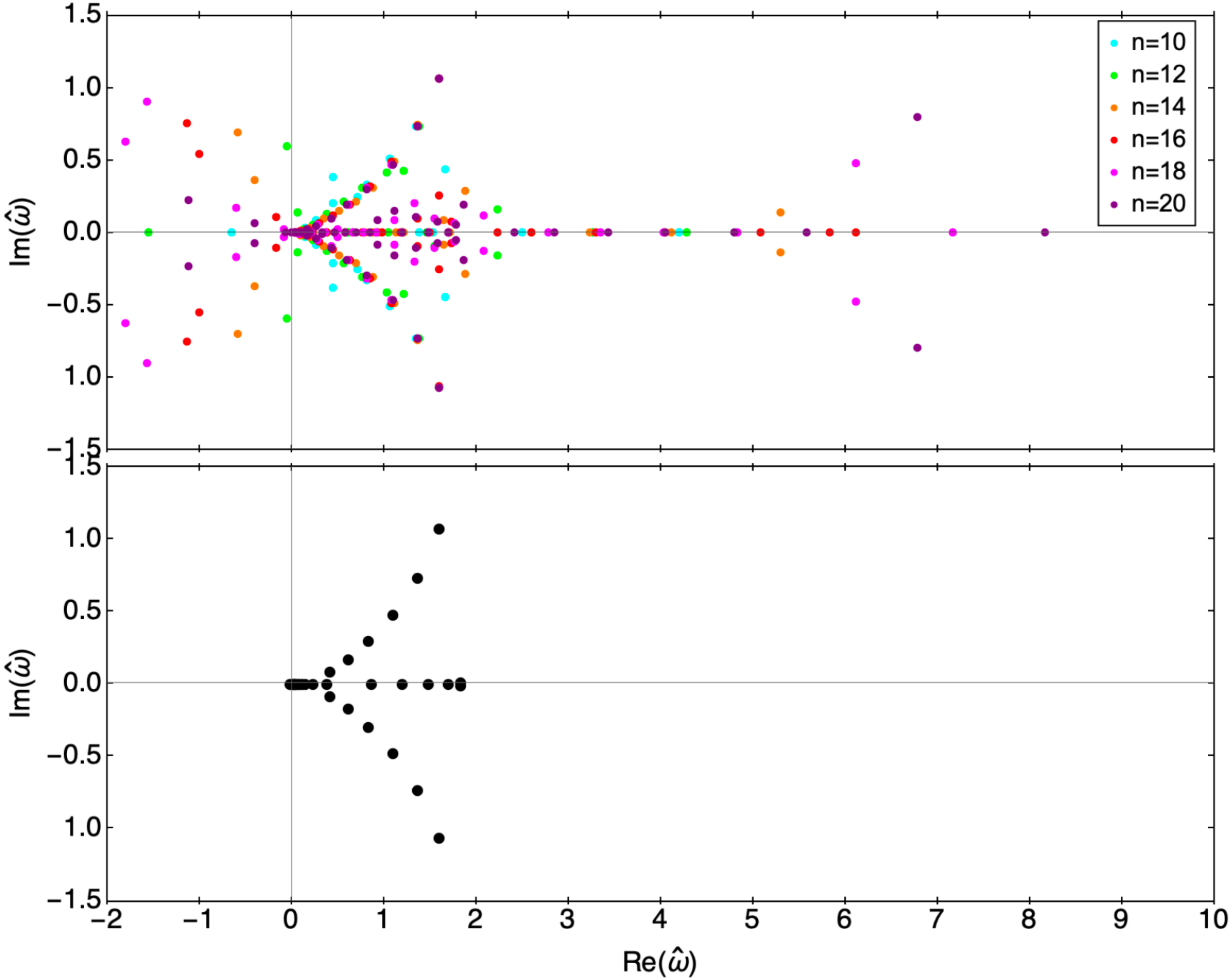
$$\times \int_{-1}^{1} \mathrm{d}\xi P_l^{|m|}(\xi) \left(\frac{1-\xi}{2}\right)^{5/4} \left(\frac{1+\xi}{2}\right) \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\left(\frac{1-\xi}{2}\right)^{1/2} P_n^{|m|}(\xi) \right],$$

$$F_{ln} = 2 \int_{-1}^{1} \mathrm{d}\xi P_l^{|m|}(\xi) \widehat{\Omega}(\xi) P_n^{|m|}(\xi),$$

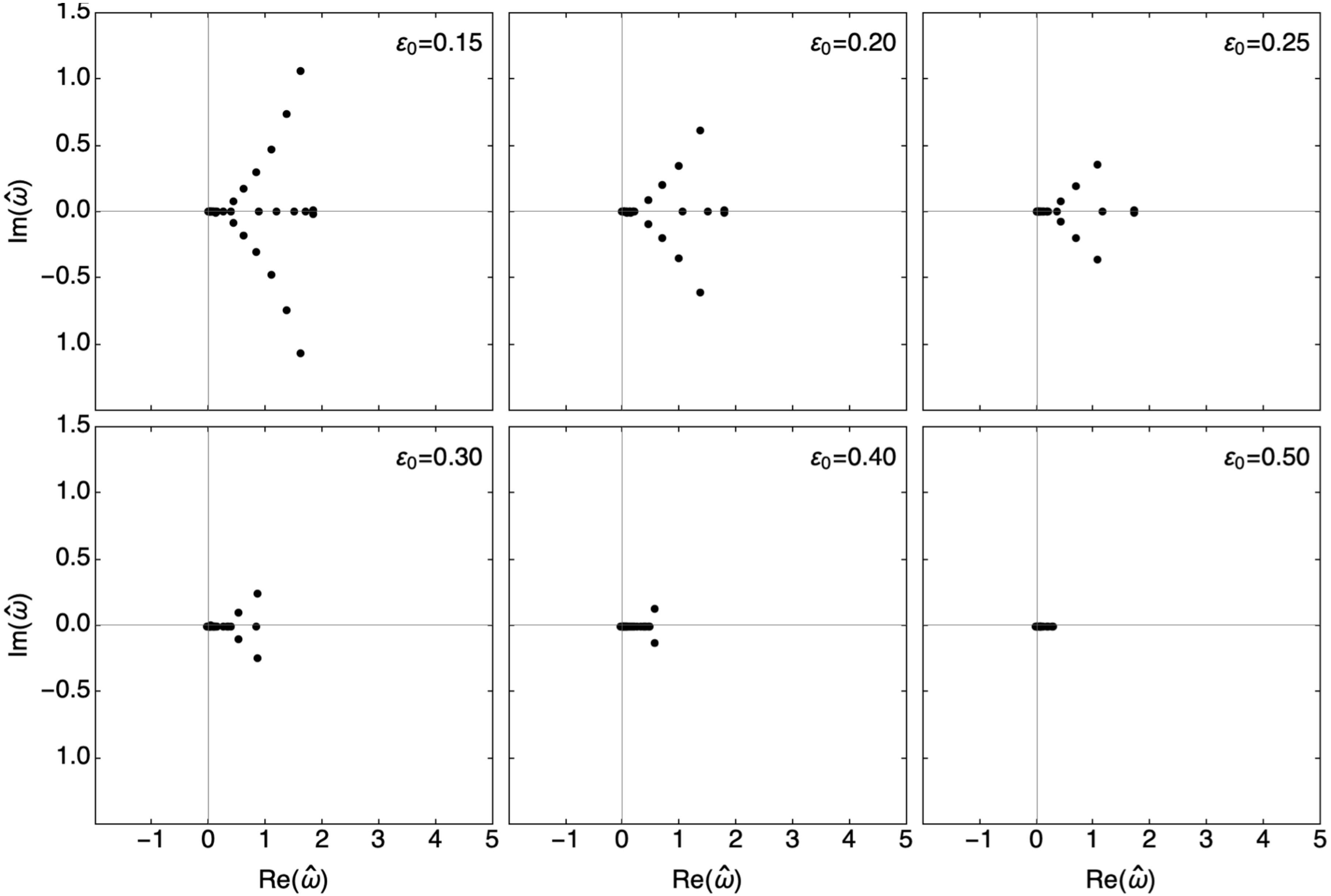
$$G_{ln} = -|m|\sqrt{1-p} \left(\frac{1}{2n+1} - \frac{\varepsilon_0}{3} \frac{1}{(1-p)^{2/3}} \right) \int_{-1}^{1} \mathrm{d}\xi P_l^{|m|}(\xi) \left(\frac{1-\xi}{2}\right)^{3/4} P_n^{|m|}(\xi),$$

$$H_{ln} = \int_{-1}^{1} \mathrm{d}\xi P_l^{|m|}(\xi) \frac{\widehat{\kappa}^2(\xi)}{2\widehat{\Omega}(\xi)} P_n^{|m|}(\xi).$$

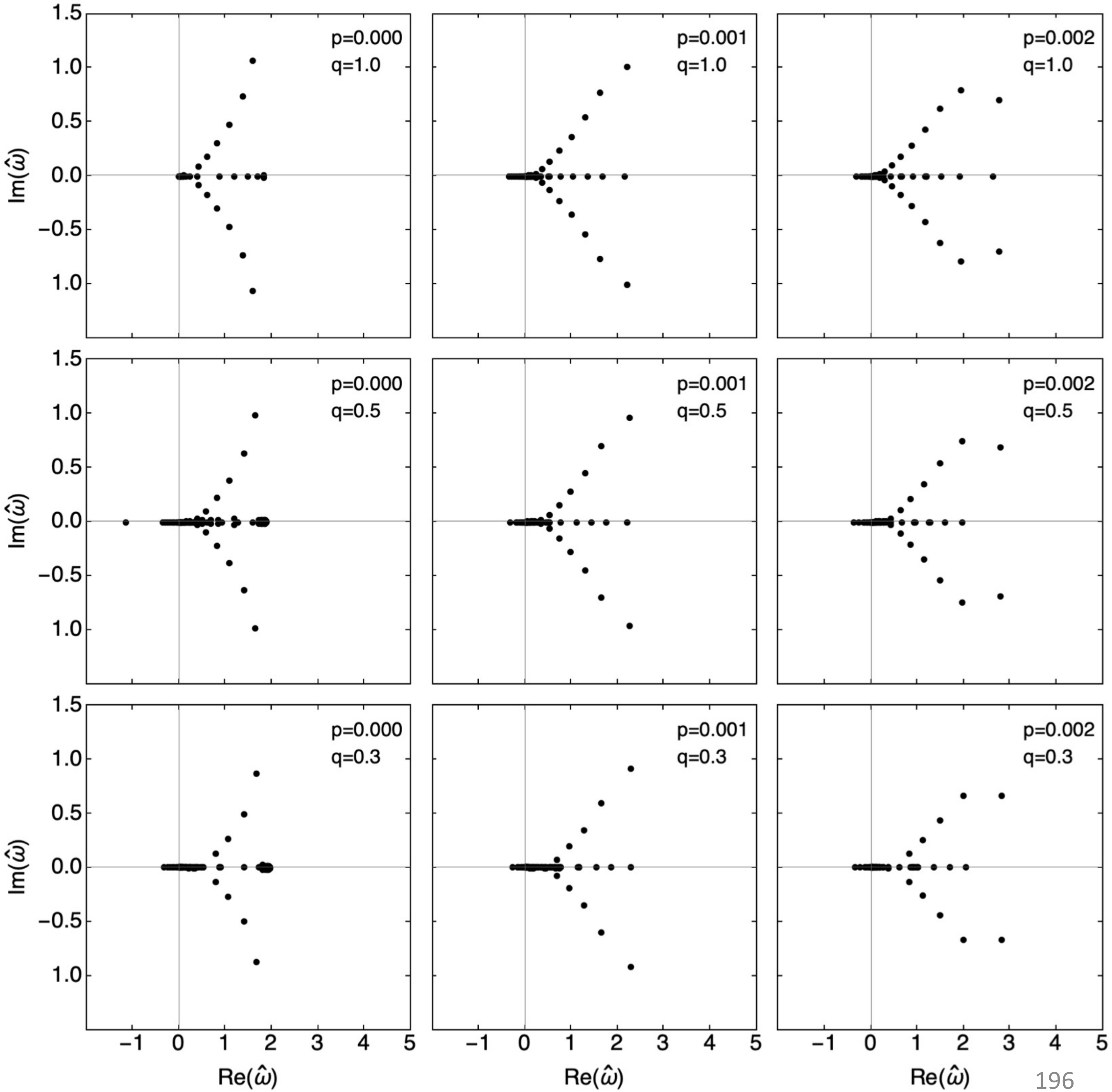
Bars: eigenvalue convergence study



Bars: eigenvalue temperature dependency



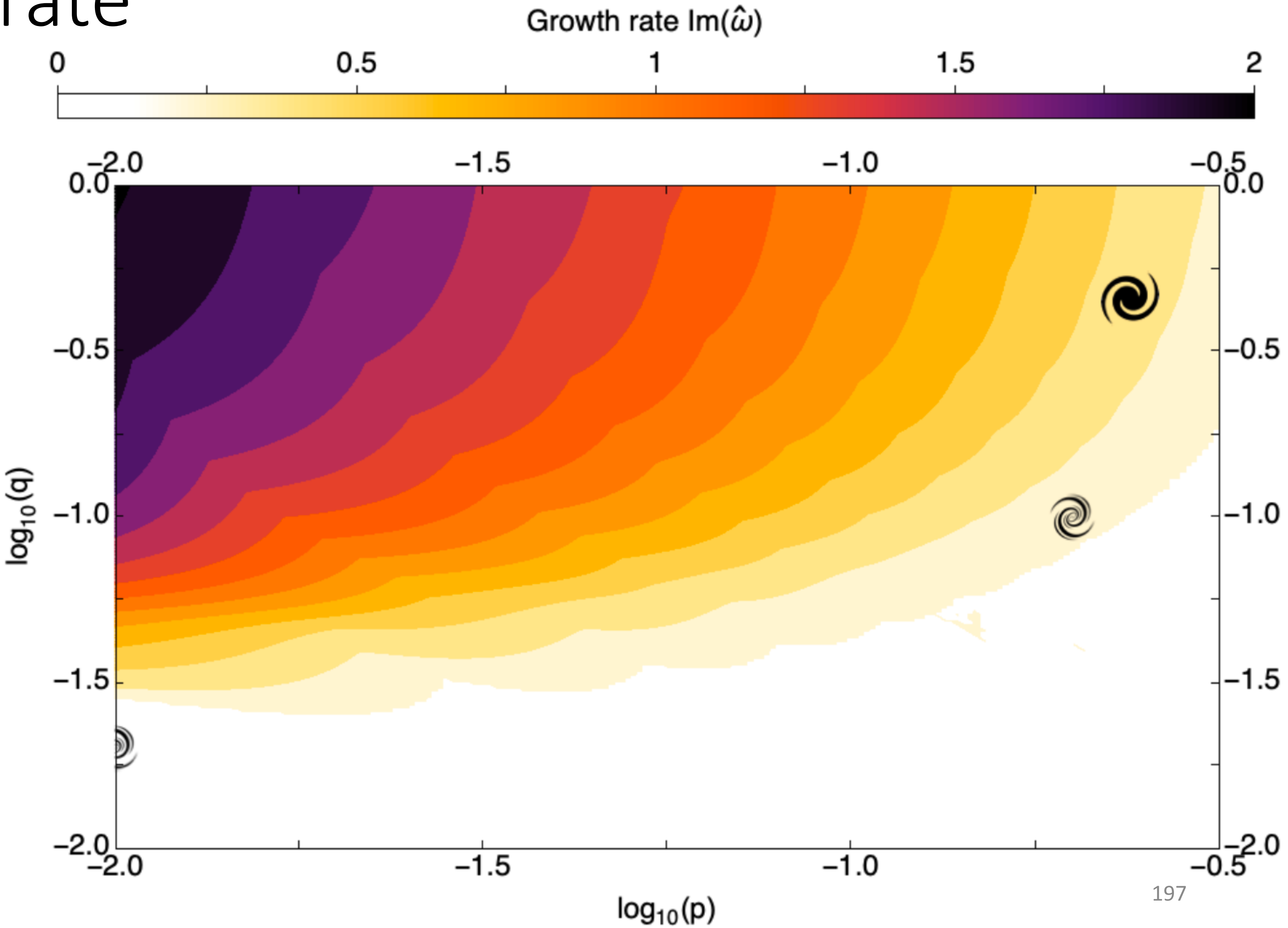
Bars: bulge/DH



Bars: growth rate

$$p = \frac{M_{\rm bulge}}{M_{\rm disc} + M_{\rm bulge}}$$

$$q = \frac{M_{\rm disc}}{M_{\rm disc} + M_{\rm halo}}$$



Secular relaxation