Kinetic theory at order $1/N^2$

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Collisional dynamics

Phase space $\mathbf{w} = (\mathbf{q}, \mathbf{p})$

Total Hamiltonian $$H = \sum_{i=1}^{N} m\, U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m^2\, U(\mathbf{w}_i, \mathbf{w}_j)$$

Scaling $m = \dfrac{M_{\mathrm{tot}}}{N}; \qquad \dfrac{1}{N} \ll 1$

Plasma

$$\begin{cases} U_{\mathrm{ext}} = \mathbf{v}^2/2 \\ U = 1/|\mathbf{x} - \mathbf{x}'| \\ \Phi_0 = 0 \\ \mathbf{w} = (\mathbf{x}, \mathbf{v}) \\ F = F(\mathbf{v}, t) \end{cases}$$

Galaxy

$$\begin{cases} U_{\mathrm{ext}} = \mathbf{v}^2/2 \\ U = -G/|\mathbf{x} - \mathbf{x}'| \\ \Phi_0 = \Phi_0(\mathbf{x}) \\ \mathbf{w} = (\boldsymbol{\theta}, \mathbf{J}) \\ F = F(\mathbf{J}, t) \end{cases}$$

Balescu-Lenard equation

Homogeneous Balescu-Lenard equation

$U_{\mathbf{k}\mathbf{k}} = 1/|\mathbf{k}|^2$

$$\frac{\partial F(\mathbf{v})}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{v}}\cdot\bigg[\int \mathrm{d}\mathbf{k}\,\mathbf{k}\int \mathrm{d}\mathbf{v}'\,\delta_{\mathrm{D}}[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\,\frac{|U_{\mathbf{k}\mathbf{k}}|^2}{|\epsilon_{\mathbf{k}}(\mathbf{k}\cdot\mathbf{v})|^2} \times\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}'}\right)F(\mathbf{v})\,F(\mathbf{v}')\bigg]$$

Balescu-Lenard equation

Homogeneous Balescu-Lenard equation

$U_{\mathbf{kk}} = 1/|\mathbf{k}|^2$

$$\frac{\partial F(\mathbf{v})}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{v}}\cdot\Bigg[\int \mathrm{d}\mathbf{k}\,\mathbf{k}\int \mathrm{d}\mathbf{v}'\,\delta_{\mathrm{D}}[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\,\frac{|U_{\mathbf{kk}}|^2}{|\epsilon_{\mathbf{k}}(\mathbf{k}\cdot\mathbf{v})|^2} \times\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}'}\right)F(\mathbf{v})\,F(\mathbf{v}')\Bigg]$$

Inhomogeneous Balescu-Lenard equation

Orbital frequencies

Dressed coupling

$$\frac{\partial F(\mathbf{J})}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}}\cdot\Bigg[\sum_{\mathbf{k},\mathbf{k}'}\mathbf{k}\int \mathrm{d}\mathbf{J}'\,\delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J}')]\,|U^{\mathrm{d}}_{\mathbf{kk}'}(\mathbf{J},\mathbf{J}')|^2 \times\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right)F(\mathbf{J})\,F(\mathbf{J}')\Bigg]$$

Short notations and 1D

Inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial J}\left[\sum_{k_1} k_1 \int \mathrm{d}J_1 \,|\, U_{k_1}(\mathbf{J}) \,|^2\, \delta_{\mathrm{D}}\big(\mathbf{k}\cdot\mathbf{\Omega}\big)\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_2(\mathbf{J})\right]$$

Two-body interactions

$$\mathbf{J} = (J_1, J_2); \qquad \mathbf{\Omega} = (\Omega[J], \Omega[J_1]);$$
$$\mathbf{k} = (k_1, -k_1); \quad F_2(\mathbf{J}) = F(J_1)\, F(J_2)$$

Interaction potential

Bare coupling coefficients

$$U(w, w') = \sum_k U_k(J, J')\, \mathrm{e}^{\mathrm{i}k(\theta-\theta')}$$

Collective effects

Inhomogeneous Balescu-Lenard equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial J}\left[\sum_{k_1} k_1 \int \mathrm{d}J_1 \, |U^{\mathrm{d}}_{k_1}(\mathbf{J})|^2 \, \delta_{\mathrm{D}}\big(\mathbf{k}\cdot\mathbf{\Omega}\big)\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_2(\mathbf{J})\right]$$

$\mathbf{J} = (J, J_1); \quad \mathbf{k} = (k_1, -k_1); \quad F_2(\mathbf{J}) = F(J)F(J_1)$

Dressed interactions

$$|U_{k_1}(\mathbf{J})|^2 \to |U^{\mathrm{d}}_{k_1}[F](\mathbf{J})|^2$$

Dynamical temperature

$$U_k(J) \propto G$$

Bare coupling

$$U^{\mathrm{d}}_k(J) \propto \frac{U_k(J)}{\epsilon(\omega)} \propto \frac{G}{1-G}$$

Dressed coupling

Dynamically hot limit

$$G \to 0$$

Relaxation time

$$T_{\mathrm{relax}} \propto T_{\mathrm{dyn}}\, N/G^2$$

1/N kinetic blocking

1/N kinetic equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial J}\left[\sum_{k_1} k_1 \int \mathrm{d}J_1 \, |U^{\mathrm{d}}_{k_1}(\mathbf{J})|^2 \, \delta_{\mathrm{D}}\big(\mathbf{k}\cdot\boldsymbol{\Omega}\big)\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_2(\mathbf{J})\right]$$

$\mathbf{J} = (J, J_1)$; $\mathbf{k} = (k_1, -k_1)$; $F_2(\mathbf{J}) = F(J)F(J_1)$

Monotonic frequency profile

$$J \mapsto \Omega(J) \quad \text{monotonic}$$

Local resonances

$$\mathbf{k}\cdot\boldsymbol{\Omega} = 0 \;\Rightarrow\; k_1\Omega(J) = k_1\Omega(J_1) \;\Rightarrow\; J_1 = J$$

Vanishing of the **crossed term**

$$\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_2(\mathbf{J}) = 0$$

Vanishing of the flux

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N}\times 0$$

This is a **kinetic blocking**

1/N kinetic blocking

1/N kinetic equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial J}\left[\sum_{k_1} k_1 \int \mathrm{d}J_1 \,|\, U^{\mathrm{d}}_{k_1}(\mathbf{J}) \,|^2\, \delta_{\mathrm{D}}\big(\mathbf{k}\cdot\mathbf{\Omega}\big)\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_2(\mathbf{J})\right]$$

$\mathbf{J} = (J, J_1)$; $\mathbf{k} = (k_1, -k_1)$; $F_2(\mathbf{J}) = F(J)F(J_1)$

Monotonic frequency profile

$$J \mapsto \Omega(J) \quad \text{monotonic}$$

Local resonances

$$\mathbf{k}\cdot\mathbf{\Omega} = 0 \;\Rightarrow\; k_1\Omega(J) = k_1\Omega(J_1) \;\Rightarrow\; J_1 = J$$

The **blocking** occurs whatever

$U(\mathbf{w}, \mathbf{w}')$

Interaction potential

$F(\mathbf{J})$

(Stable) DF

$|\, U_k(\mathbf{J}) \,|^2, \; |\, U^{\mathrm{d}}_k(\mathbf{J}) \,|^2$

Hot/Cold limit

How to derive a $1/N^2$ kinetic equation?

How to derive a $1/N$ kinetic equation?

Balescu-Lenard from BBGKY

N **identical particles** of mass $m = M_{\mathrm{tot}}/N$ in phase space $\mathbf{w}_i = (\mathbf{q}_i, \mathbf{p}_i)$

Total **Hamiltonian**

$$H_N = \sum_{i=1}^{N} m\, U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m^2\, U(\mathbf{w}_i, \mathbf{w}_j)$$

3D self-gravitating systems

$$U_{\mathrm{ext}} = \mathbf{v}^2/2$$
$$U = -G/|\mathbf{x} - \mathbf{x}'|$$

System characterised by the **N-body PDF** $P_N(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$

Continuity equation in phase space

$$\frac{\partial P_N}{\partial t} + \sum_i \frac{\partial}{\partial \mathbf{w}_i} \cdot \left(P_N\, \dot{\mathbf{w}}_i \right) = 0$$

Exact **Liouville equation**

$$\frac{\partial P_N}{\partial t} + \left[P_N, H_N \right]_N = 0$$

Poisson bracket

BBGKY hierarchy

Reduced DFs

$$F_n(\mathbf{w}_1, \ldots, \mathbf{w}_n, t) = m^n \frac{N!}{(N-n)!} \int \mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \, P_N(\mathbf{w}_1, \ldots, \mathbf{w}_N, t)$$

BBGKY hierarchy

$$\frac{\partial F_n}{\partial t} + \left[F_n, H_n\right]_n + \int \mathrm{d}\mathbf{w}_{n+1} \left[F_{n+1}, \delta H_{n+1}\right]_n = 0$$

With

$$H_n = \sum_{i=1}^{n} U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{n} m \, U(\mathbf{w}_i, \mathbf{w}_j)$$

Isolated n-body system

$$\delta H_{n+1} = \sum_{i=1}^{n} U(\mathbf{w}_i, \mathbf{w}_{n+1})$$

Interactions with (n+1)

BBGKY at 1/N

Cluster representation

$$\begin{cases} F_2(\mathbf{w},\mathbf{w}') = F_1(\mathbf{w})\,F_1(\mathbf{w}') + G_2(\mathbf{w},\mathbf{w}') \\ F_3(\mathbf{w},\mathbf{w}',\mathbf{w}'') = \ldots + G_3(\mathbf{w},\mathbf{w}',\mathbf{w}'') \end{cases} \Longrightarrow \begin{cases} G_2 \sim 1/N \\ G_3 \sim 1/N^2 \end{cases}$$

Truncation at **order 1/N**: 2 dynamical quantities

$F(\mathbf{w},t)$ 1-body DF

$G(\mathbf{w},\mathbf{w}',t)$ 2-body correlation

BBGKY - 1

$$\frac{\partial F}{\partial t} + \big[F, H_0\big]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}' \, \big[G, U(\mathbf{w},\mathbf{w}')\big]_{\mathbf{w}} = 0$$

BBGKY - 2

$$\frac{\partial G}{\partial t} + \big[G, H_0\big]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}''\, G(\mathbf{w}',\mathbf{w}'')\, \big[F(\mathbf{w}), U(\mathbf{w},\mathbf{w}'')\big]_{\mathbf{w}}$$

$$+ m\, \big[F(\mathbf{w})\,F(\mathbf{w}'), U(\mathbf{w},\mathbf{w}')\big]_{\mathbf{w}} + (\mathbf{w} \leftrightarrow \mathbf{w}') = 0$$

BBGKY - 1

$$\frac{\partial F}{\partial t} + \left[F, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}} = 0$$

$\left[F, H_0\right]_{\mathbf{w}}$ Mean-field advection

$\int \mathrm{d}\mathbf{w}' \left[G, U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}}$ Collision term

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BBGKY - 2

$$\frac{\partial G}{\partial t} + \left[G, H_0\right]_{\mathbf{w}} + \int \mathrm{d}\mathbf{w}'' \, G(\mathbf{w}', \mathbf{w}'') \left[F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')\right]_{\mathbf{w}}$$

$$+ m \left[F(\mathbf{w}) \, F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}} + (\mathbf{w} \leftrightarrow \mathbf{w}') = 0$$

$\left[G, H_0\right]_{\mathbf{w}}$ Mean-field advection

$\int \mathrm{d}\mathbf{w}'' \, G(\mathbf{w}', \mathbf{w}'') \left[F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')\right]_{\mathbf{w}}$ Collective effects

$m \left[F(\mathbf{w}) \, F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')\right]_{\mathbf{w}}$ 1-body DF sourcing

How to solve BBGKY

Adiabatic approximation

i.e. evolution along **quasi-stationary states**

$$F = F(\mathbf{J}, t) \;\; ; \;\; H_0 = H_0(\mathbf{J}, t) \quad \Longrightarrow \quad \big[F_0(\mathbf{J}) \,, H_0(\mathbf{J})\big]_{\mathbf{w}} = 0$$

Mean-field equilibrium

Timescale separation

$$\frac{\partial G}{\partial t} + \big[G, H_0\big]_{\mathbf{w}} + (\ldots) = 0$$

$$\frac{\partial F}{\partial t} = - \int \mathrm{d}\mathbf{w}' \, \big[G, U(\mathbf{w}, \mathbf{w}')\big]_{\mathbf{w}}$$

Collision operator

$$\longrightarrow \begin{cases} T_G \simeq T_{\mathrm{dyn}} \\ T_F \simeq N \times T_G \end{cases}$$

Bogoliubov's Ansatz

$$\frac{\partial G}{\partial t} = \mathrm{BBGKY}_2\big[F = \mathrm{cst}\,, G\big]$$

$$\frac{\partial F}{\partial t} = \mathrm{BBGKY}_1\big[F\,, G(t \rightarrow +\infty)\big]$$

How to derive
a $1/N^2$ kinetic equation?

How to derive a $1/N^2$ kinetic equation in the hot limit?

BBGKY at order 1/N²

Describing the system

$$F(\mathbf{w}_1) \propto 1$$

1-body DF

$$G_2(\mathbf{w}_1, \mathbf{w}_2) \propto 1/N$$

2-body correlation

$$G_3(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \propto 1/N^2$$

3-body correlation

Correlation **splitting**

$$G_2 = \frac{1}{N}\, G_2^{(1)} + \frac{1}{N^2}\, G_2^{(2)}$$

Rocha-Filho+2015

Neglecting **collective effects**

$$\int \mathrm{d}w_3\, G_2^{(1)}(\mathbf{w}_2, \mathbf{w}_3)\, U(\mathbf{w}_1, \mathbf{w}_3) \to 0$$

Hot limit

Neglecting dynamically cold terms

$$G_2^{(1)} \times G_2^{(1)} \to 0 \;\; \text{in} \;\; \partial G_3 / \partial t$$

Hot limit

A well-posed hierarchy

Solving in **sequence**

$$\frac{\partial G_2^{(1)}}{\partial t} + \left[G_2^{(1)}, H_0\right] = S[F]$$

(2 terms in the rhs)

$$\frac{\partial G_3}{\partial t} + \left[G_3, H_0\right] = S[F, G_2^{(1)}]$$

(24 terms in the rhs)

$$\frac{\partial G_2^{(2)}}{\partial t} + \left[G_2^{(2)}, H_0\right] = S[F, G_3]$$

(2 terms in the rhs)

$$\frac{\partial F}{\partial t} = C[G_2^{(2)}]$$

(4 terms in the rhs)

After **re-injecting** and **expanding** all the derivatives

~1,000 terms

Simplifying the collision operator

Interaction potential

$$U(\mathbf{w}, \mathbf{w}') = \sum_k U_k(J, J')\, \mathrm{e}^{\mathrm{i}k(\theta-\theta')}$$

Monotonic frequency profile

$$J \mapsto \Omega(J)$$

Large time limit

$$\lim_{t\to+\infty} \int_0^t \mathrm{d}t'\, \mathrm{e}^{\mathrm{i}(t-t')\omega_{\mathrm{R}}} = \pi\delta_{\mathrm{D}}(\omega_{\mathrm{R}}) + \mathrm{i}\,\mathscr{P}\left(\frac{1}{\omega_{\mathrm{R}}}\right)$$

Number of terms keeps growing: ~10,000 terms

Do NOT perform these calculations by hand!

Using a custom grammar in **Mathematica**

Simplifying the collision operator

Typical shape

$$\frac{\partial F(J)}{\partial t} = \frac{\partial}{\partial J}\left[\sum_{k_1,k_2}\int \mathrm{d}J_1\mathrm{d}J_2 \ldots\right]$$

Relabellings

$$\{J_1, J_2, k_1, k_2\} \quad \to \quad \omega_{\mathrm{R}} = (k_1+k_2)\Omega[J] - k_1\Omega[J_1] - k_2\Omega[J_2]$$

Integration by parts

$$\delta'_{\mathrm{D}} \;\to\; \delta_{\mathrm{D}}; \quad \partial^2_{J_1}F \;\to\; \partial_{J_1}F$$

Scaling relations

$$\delta_{\mathrm{D}}[\alpha\omega_{\mathrm{R}}] = \delta_{\mathrm{D}}[\omega_{\mathrm{R}}]/|\alpha|$$

Monotonic frequency profile

$$\int \mathrm{d}J_2\, f(J_1, J_2)\, \delta_{\mathrm{D}}(\Omega[J_1] - \Omega[J_2]) = f(J_1, J_1)/|\Omega'[J_1]|$$

Final equation

1D **1/N²** inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = 2\pi^3 m^2 \frac{\partial}{\partial J}\Bigg[\sum_{k_1,k_2} \frac{k_1+k_2}{k_1^2(k_1+k_2)^2} \mathscr{P}\!\int \frac{\mathrm{d}J_1}{(\Omega[J]-\Omega[J_1])^4} \times \int \mathrm{d}J_2\, \mathscr{U}_{k_1k_2}(\mathbf{J})\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_3(\mathbf{J})\Bigg]$$

The DF appears three times

$$\mathbf{J} = \big(J, J_1, J_2\big); \quad F_3(\mathbf{J}) = F(J)\, F(J_1)\, F(J_2)$$

Three-body resonances

$$\mathbf{k} = \big(k_1+k_2, -k_1, -k_2\big); \quad \mathbf{\Omega} = \big(\Omega[J], \Omega[J_1], \Omega[J_2]\big)$$

Short notations

1/N inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial J}\left[\sum_{k_1} k_1 \int \mathrm{d}J_1 \, |U_{k_1}(J)|^2 \, \delta_{\mathrm{D}}\big(\mathbf{k}\cdot\mathbf{\Omega}\big) \, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_2(\mathbf{J})\right]$$

$\mathbf{J} = (J, J_1);\quad \mathbf{k} = (k_1, -k_1);\quad F_2(\mathbf{J}) = F(J)F(J_1)$

1/N² inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N^2}\frac{\partial}{\partial J}\left[\sum_{k_1,k_2} (k_1+k_2) \int \mathrm{d}J_1 \mathrm{d}J_2 \, |\Lambda_{\mathbf{k}}(\mathbf{J})|^2 \, \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}) \, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_3(\mathbf{J})\right]$$

$\mathbf{J} = (J, J_1, J_2);\quad \mathbf{k} = (k_1+k_2, -k_1, -k_2);\quad F_3(\mathbf{J}) = F(J)F(J_1)F(J_2)$

The two equations are **strikingly similar**

What are the properties of the $1/N^2$ kinetic equation?

Coupling coefficient

Kinetic equation

$$\frac{\partial F(J)}{\partial t} = 2\pi^3 m^2 \frac{\partial}{\partial J}\left[\sum_{k_1,k_2} \frac{k_1+k_2}{k_1^2(k_1+k_2)^2}\mathscr{P}\int \frac{\mathrm{d}J_1}{(\Omega[J]-\Omega[J_1])^4} \times \int \mathrm{d}J_2\, \mathcal{U}_{k_1k_2}(\mathbf{J})\,\delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}F_3(\mathbf{J})\right]$$

$\mathbf{J}=(J,J_1,J_2)$; $\mathbf{k}=(k_1+k_2,-k_1,-k_2)$; $F_3(\mathbf{J})=F(J)F(J_1)F(J_2)$

Interaction potential

$$U(\mathbf{w},\mathbf{w}') = \sum_k U_k[J,J']\,\mathrm{e}^{\mathrm{i}k(\theta-\theta')}$$

Coupling coefficient

$$\mathcal{U}_{k_1k_2}(\mathbf{J}) = \left[\left(\Omega[J]-\Omega[J_1]\right)\mathcal{U}^{(1)}_{k_1k_2}(\mathbf{J}) + k_2\,\mathcal{U}^{(2)}_{k_1k_2}(\mathbf{J})\right]^2$$

Coupling coefficient

Three-body coupling

$$\mathcal{U}_{k_1k_2}(\mathbf{J}) = \Big[\Big(\Omega[J]-\Omega[J_1]\Big)\,\mathcal{U}^{(1)}_{k_1k_2}(\mathbf{J}) + k_2\,\mathcal{U}^{(2)}_{k_1k_2}(\mathbf{J})\Big]^2$$

Two parts

$$\begin{aligned}
\mathcal{U}^{(1)}_{k_1k_2}(\mathbf{J}) = {} & k_2(k_1+k_2)\,\Big\{U_{k_1+k_2}(J,J_2)\,\partial_{J_2}U_{k_1}(J_1,J_2) - U_{k_2}(J,J_2)\,\partial_J U_{k_1}(J,J_1)\Big\} \\
& + k_1(k_1+k_2)\,\Big\{U_{k_1}(J,J_1)\,\partial_J U_{k_2}(J,J_2) - U_{k_1+k_2}(J,J_1)\,\partial_{J_1}U_{k_2}(J_1,J_2)\Big\} \\
& - k_1k_2\,\Big\{U_{k_2}(J_1,J_2)\,\partial_{J_1}U_{k_1+k_2}(J,J_1) - U_{k_1}(J_1,J_2)\,\partial_{J_2}U_{k_1+k_2}(J,J_2)\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{U}^{(2)}_{k_1k_2}(\mathbf{J}) = {} & (k_1+k_2)\,\frac{\mathrm{d}\Omega}{\mathrm{d}J}\,U_{k_1}(J,J_1)\,U_{k_2}(J,J_2) \\
& - k_1\,\frac{\mathrm{d}\Omega}{\mathrm{d}J_1}\,U_{k_1+k_2}(J,J_1)\,U_{k_2}(J_1,J_2) \\
& - k_2\,\frac{\mathrm{d}\Omega}{\mathrm{d}J_2}\,U_{k_1}(J_1,J_2)\,U_{k_1+k_2}(J,J_2)
\end{aligned}$$

Properties

Conservation laws

$$\begin{cases} M(t) = \int \mathrm{d}J\, F(J,t) & \text{(total mass)} \\ P(t) = \int \mathrm{d}J\, J\, F(J,t) & \text{(total momentum)} \\ E(t) = \int \mathrm{d}J\, H_0(J)\, F(J,t) & \text{(total energy)} \end{cases}$$

Coupling amplitude

$$U_k(J) \propto G$$

Bare coupling

$$\Lambda_{\mathbf{k}}(\mathbf{J}) \propto |\, U_k(\mathbf{J})\,|^2$$

Three-body resonances

$$\frac{\partial F}{\partial t} \propto \frac{1}{N^2}\,|\, \Lambda_{\mathbf{k}}(\mathbf{J})\,|^2$$

Kinetic equation

Relaxation time

$$T_{\mathrm{relax}} \propto T_{\mathrm{dyn}}\, N^2 / G^4$$

Well-defined

High-order **resonant denominator**

$$\frac{\partial F(J)}{\partial t} \propto \mathscr{P}\int \mathrm{d}J_1 \, \frac{K(J,J_1)}{(\Omega[J]-\Omega[J_1])^4}$$
$$\propto \mathscr{P}\int \mathrm{d}\Omega_1 \, \frac{K(\Omega,\Omega_1)}{(\Omega-\Omega_1)^4}$$

Symmetrisation using **fundamental resonances**

$$K(\Omega,\Omega+\delta\Omega) = \mathscr{O}\left[(\delta\Omega)^3\right]$$

Most deft calculation

Principal value is **well-defined**

$$\mathscr{P}\int \mathrm{d}\Omega_1 \, \frac{K(\Omega,\Omega_1)}{(\Omega-\Omega_1)^4} \simeq \mathscr{P}\int \frac{\mathrm{d}\delta\Omega}{\delta\Omega}$$

H-Theorem

Boltzmann **entropy**

$$S(t) = -\int \mathrm{d}J\, s\big[F(J,t)\big] \quad \text{with} \quad s(F) = F\ln(F)$$

Rate of entropy growth

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\,|\,\Lambda_{\mathbf{k}}(\mathbf{J})\,|^2 \left\{(k_1+k_2)\frac{F'(J)}{F(J)} - k_1\frac{F'(J_1)}{F(J_1)} - k_2\frac{F'(J_2)}{F(J_2)}\right\}^2$$

H-Theorem

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geq 0$$

Steady states

Boltzmann DF

$$F_{\mathrm{B}}(J) \propto \mathrm{e}^{-\beta H_0(J)+\gamma J} \longrightarrow \frac{\partial F_{\mathrm{B}}}{\partial t} \propto -\beta\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\, (\mathbf{k}\cdot\mathbf{\Omega}) = 0$$

Steady states

Boltzmann DF

$$F_{\mathrm{B}}(J) \propto \mathrm{e}^{-\beta H_0(J)+\gamma J} \longrightarrow \frac{\partial F_{\mathrm{B}}}{\partial t} \propto -\beta\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\,(\mathbf{k}\cdot\mathbf{\Omega}) = 0$$

Constraint from **H-Theorem**

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\,|\Lambda_{\mathbf{k}}(\mathbf{J})|^2 \left\{(k_1+k_2)\frac{F'(J)}{F(J)} - k_1\frac{F'(J_1)}{F(J_1)} - k_2\frac{F'(J_2)}{F(J_2)}\right\}^2$$

Line constraint (with a non-vanishing coupling)

$$G(\Omega) = \frac{F'(J[\Omega])}{F(J[\Omega])} \;\Rightarrow\; \forall \Omega_1, \Omega_2 : G\left(\frac{k_1\Omega_1+k_2\Omega_2}{k_1+k_2}\right) = \frac{k_1 G(\Omega_1)+k_2 G(\Omega_2)}{k_1+k_2}$$

Recovering the **Boltzmann DF**

$$\frac{F'(J)}{F(J)} = -\beta\Omega(J)+\gamma \;\Rightarrow\; F(J) \propto \mathrm{e}^{-\beta H_0(J)+\gamma J}$$

Does it match with
N-body simulations?

Does it work?

Spin dynamics

$$H = \sum_{i=1}^{N} m\, U_{\text{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m^2\, U(\mathbf{w}_i, \mathbf{w}_j)$$

Interaction potentials

Classical Heisenberg spins

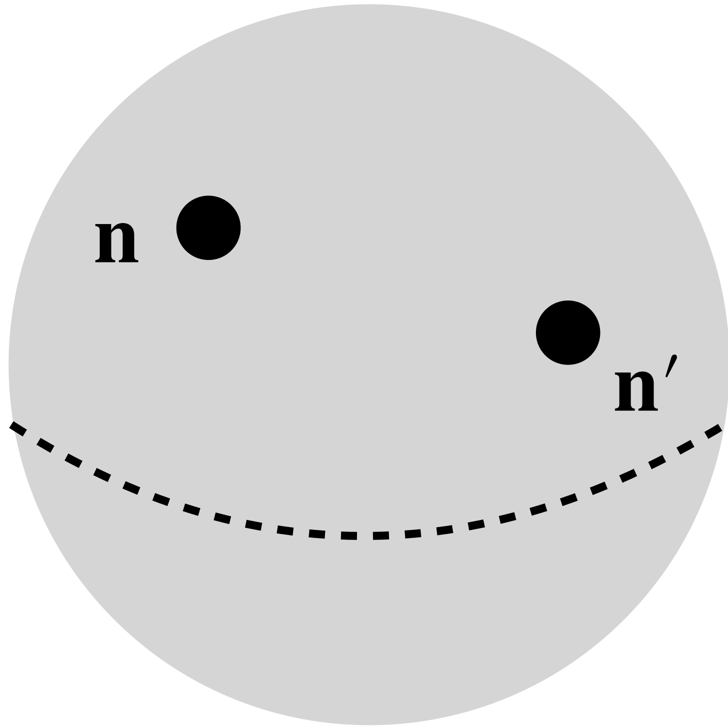
$$U_{\text{ext}}(\mathbf{w}) \propto \left(\mathbf{n}\cdot\hat{\mathbf{z}}\right)^2; \quad U(\mathbf{w}, \mathbf{w}') \propto \mathbf{n}\cdot\mathbf{n}'$$

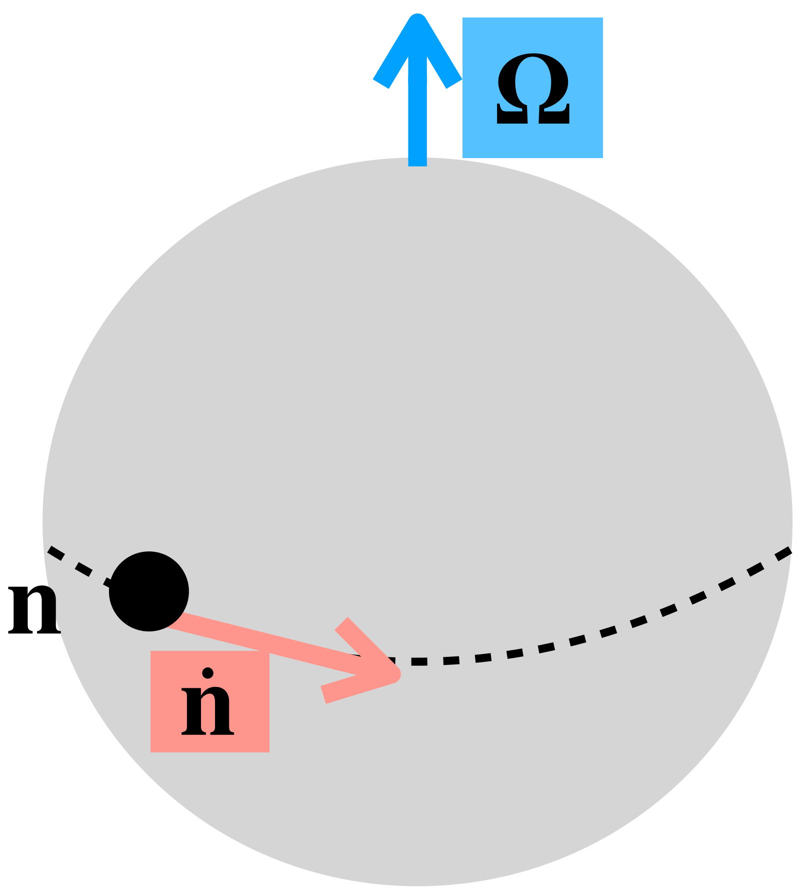
Dynamics on the **unit sphere**

$$\dot{\mathbf{n}} = \frac{\partial H}{\partial \mathbf{n}} \times \mathbf{n}$$

Simulations using **rotations**

$$|\mathbf{n}| = 1$$

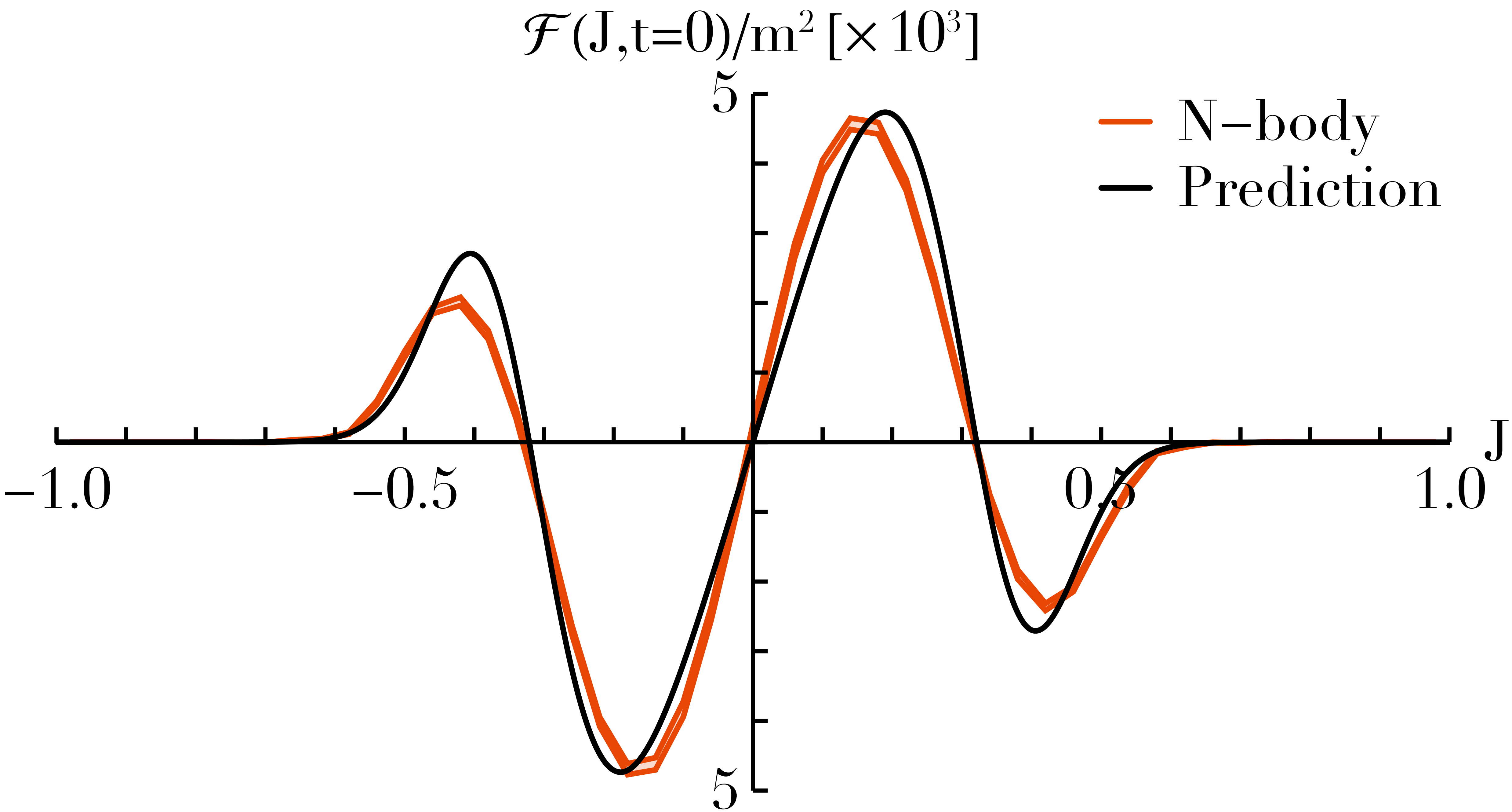




Does it match?

Diffusion flux

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial J}\mathscr{F}(J)$$



Can one block the $1/N^2$ relaxation?

Second-order kinetic blocking

Constraint from the **H-Theorem**

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\, |\Lambda_{\mathbf{k}}(\mathbf{J})|^2 \left\{ (k_1+k_2)\frac{F'(J)}{F(J)} - k_1 \frac{F'(J_1)}{F(J_1)} - k_2 \frac{F'(J_2)}{F(J_2)} \right\}^2$$

Killing three-body resonances

$$\forall k_1, k_2, J, J_1 : \;\; |\Lambda_{k_1 k_2}(J, J_1, J_2^{\mathrm{res}})|^2 = 0$$

Simple **frequency profile**

$$\Omega[J] \propto J$$

Second-order kinetic blocking

$$U(\mathbf{w}, \mathbf{w}') \propto |J - J'|^{\alpha} \sum_{k=1}^{+\infty} \frac{1}{|k|^{\alpha}} \cos\big[k\,(\theta - \theta')\big]$$

Second-order kinetic blocking

Constraint from **H-Theorem**

$$\frac{\mathrm{d}S}{\mathrm{d}t} \propto \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\,|\,\Lambda_{\mathbf{k}}(\mathbf{J})\,|^2 \left\{(k_1+k_2)\,\frac{F'(J)}{F(J)} - k_1\frac{F'(J_1)}{F(J_1)} - k_2\,\frac{F'(J_2)}{F(J_2)}\right\}^2$$

Killing three-body resonances

$$\forall k_1, k_2, J, J_1 : \;\; |\,\Lambda_{k_1 k_2}(J, J_1, J_2^{\mathrm{res}})\,|^2 = 0$$

Simple **frequency profile**

$$\Omega[J] \propto J$$

Second-order kinetic blocking

$$U(\mathbf{w},\mathbf{w}') \propto |\,J - J'\,|^{2n}\, B_{2n}\Big[\frac{1}{2\pi}\, w_{2\pi}(\theta - \theta')\Big]$$

Bernoulli polynomial

Angle wrapping

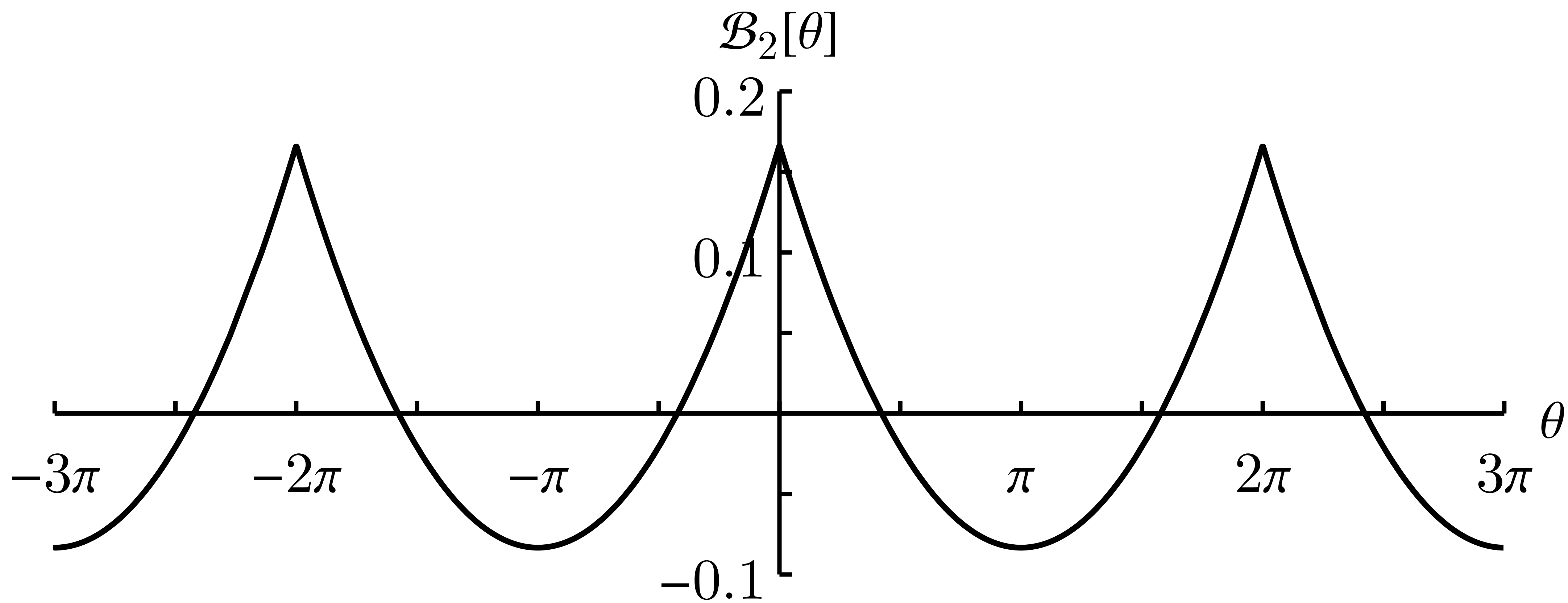
The B2 model

A **nasty** interaction

$$U(\mathbf{w}, \mathbf{w}') = G\,(J - J')^2\,\mathscr{B}_2[\theta - \theta']$$

Bernoulli polynomial

Angular dependence



Hard to simulate, because **not smooth** at crossing

Fast to simulate, because (almost) **polynomial-like**

How does the B2 system relax?

Relaxation time for the B2 model

Total **Hamiltonian**

$$H_N = \sum_{i=1}^{N} m\, U_{\mathrm{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m^2\, U(\mathbf{w}_i, \mathbf{w}_j)$$

Interaction potential

$$U(\mathbf{w}, \mathbf{w}') = G\,(J - J')^2\, \mathscr{B}_2[\theta - \theta']$$

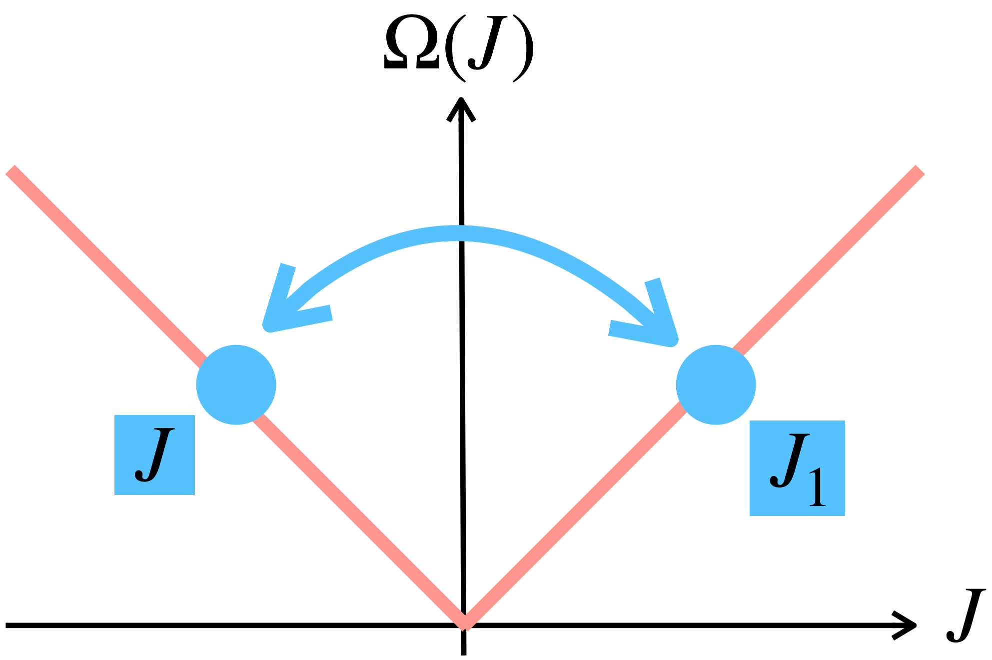
External potential

$$\Omega(J) = \Omega[U_{\mathrm{ext}}](J)$$

Different **frequency profiles**

$$\begin{cases} (1):\ \Omega(J) = J; \\ (2):\ \Omega(J) = J\,|\,J\,|\,; \\ (3):\ \Omega(J) = J; \end{cases}$$

Profile (1)



Frequency profile

$$\Omega(J) = |J|$$ is **non-monotonic**

1/N Landau equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N}\frac{\partial}{\partial J}\left[\sum_{k_1} k_1 \int \mathrm{d}J_1\, |U_{k_1}(J)|^2\, \delta_{\mathrm{D}}\big(\mathbf{k}\cdot\mathbf{\Omega}\big)\, \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} F_2(\mathbf{J})\right]$$

$\mathbf{J} = (J, J_1)$; $\mathbf{k} = (k_1, -k_1)$; $F_2(\mathbf{J}) = F(J)F(J_1)$

Non-local resonances

$$\mathbf{k}\cdot\mathbf{\Omega} = 0 \implies J_1 \neq J$$

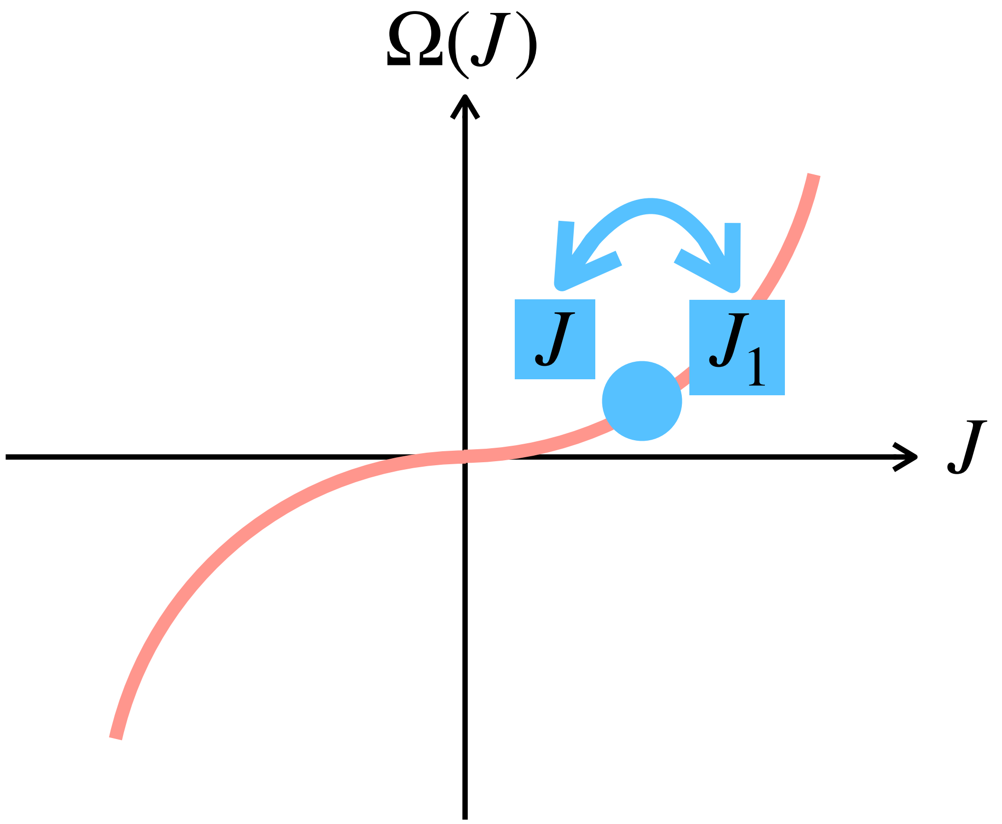
Relaxation time (hot limit)

$$T_{\mathrm{relax}} \propto T_{\mathrm{dyn}}\, N/G^2$$

Profile (2)

Frequency profile

$\Omega(J) = J\,|\,J\,|$ is **monotonic**



1/N dynamics

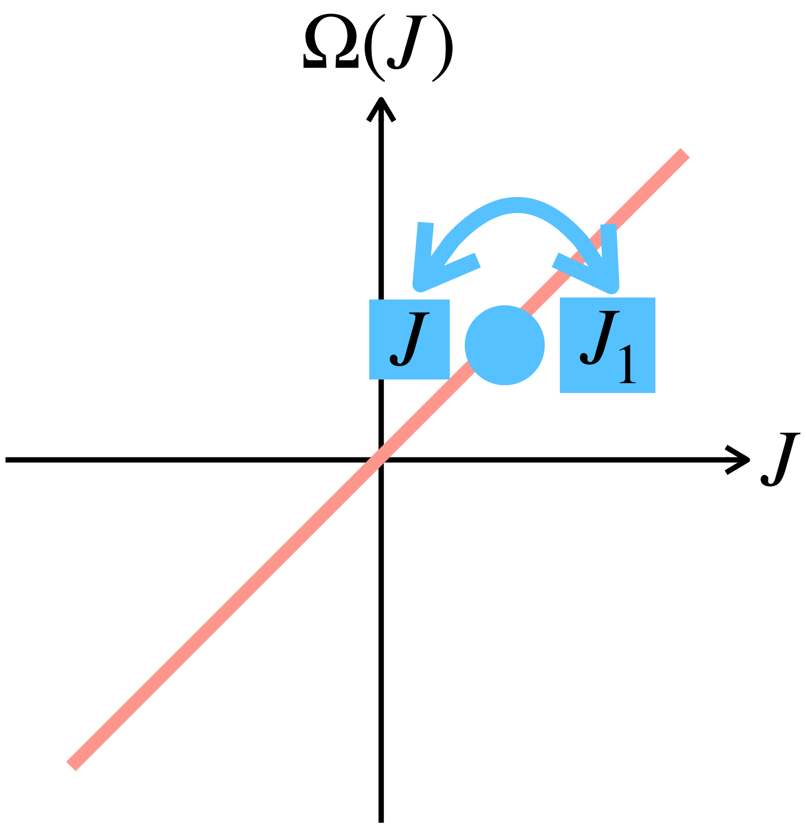
$$\mathbf{k}\cdot\mathbf{\Omega} = 0 \;\Rightarrow\; J_1 = J \;\Rightarrow\; \frac{\partial F}{\partial t} = \frac{1}{N} \times 0$$

1/N² dynamics

$$|\,\Lambda_{\mathbf{k}}(\mathbf{J}_{\mathrm{res}})\,|^2 \neq 0 \;\Rightarrow\; \frac{\partial F}{\partial t} = \frac{1}{N^2} \times \cdots$$

Relaxation time (hot limit) $T_{\mathrm{relax}} \propto T_{\mathrm{dyn}}\, N^2 / G^4$

Profile (3)



Frequency profile

$\Omega(J) = J$ is **monotonic** and **nasty**

1/N dynamics

$$\mathbf{k}\cdot\mathbf{\Omega} = 0 \;\Rightarrow\; J_1 = J \;\Rightarrow\; \frac{\partial F}{\partial t} = \frac{1}{N} \times 0$$

1/N² dynamics

$$|\Lambda_{\mathbf{k}}(\mathbf{J}_{\mathrm{res}})|^2 = 0 \;\Rightarrow\; \frac{\partial F}{\partial t} = \frac{1}{N^2} \times 0$$

What is the **relaxation time**?

Relaxation of profile (3)

1/N² inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = \frac{1}{N^2}\frac{\partial}{\partial J}\left[\sum_{k_1,k_2}\int \mathrm{d}J_1 \mathrm{d}J_2 \,|\,\Lambda_{\mathbf{k}}(\mathbf{J})\,|^2\, \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega})\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}F_3(\mathbf{J})\right]$$

$\mathbf{J} = (J, J_1, J_2);\quad \mathbf{k} = (k_1 + k_2, -k_1, -k_2);\quad F_3(\mathbf{J}) = F(J)F(J_1)F(J_2)$

1/N² inhomogeneous Balescu-Lenard equation

$$|\,\Lambda_{\mathbf{k}}(\mathbf{J})\,|^2 \rightarrow |\,\Lambda^{\mathrm{d}}_{\mathbf{k}}[F](\mathbf{J})\,|^2$$

Intricate substitution
(but unknown)

Hot limit

$$\Lambda^{\mathrm{d}}_{\mathbf{k}}[F](\mathbf{J}) \underset{G\rightarrow 0}{=} \cancel{\Lambda_{\mathbf{k}}(\mathbf{J})} + G^3\,\Lambda^{(3)}_{\mathbf{k}}[F](\mathbf{J}) + \mathcal{O}(G^4)$$

Relaxation time (hot limit)

$$T_{\mathrm{relax}} \propto T_{\mathrm{dyn}}\, N^2 / G^6$$

Four-body correlations?

Asymptotics of a **hot 1/N³** kinetic equation

$$m\,U \rightarrow G_2^{(1)} \rightarrow G_3^{(2)} \rightarrow G_4^{(3)} \rightarrow G_3^{(3)} \rightarrow G_2^{(3)} \rightarrow \partial F/\partial t$$

$$G_2 = \frac{1}{N}G_2^{(1)} + \frac{1}{N^2}G_2^{(2)} + \frac{1}{N^3}G_2^{(3)};\quad G_3 = \frac{1}{N^2}G_3^{(2)} + \frac{1}{N^3}G_3^{(3)};\quad G_4 = \frac{1}{N^3}G_4^{(3)}$$

Relaxation time for (hot) **four-body** interactions

$$T_{\mathrm{relax}}^{4-\mathrm{body}} \propto T_{\mathrm{dyn}}\, N^3 / G^6$$

``**Leaks**'' from dressed three-body interactions always dominate

$$T_{\mathrm{relax}}^{3-\mathrm{body}} \propto T_{\mathrm{dyn}}\, N^2 / G^6$$

1/N³ effects never drive relaxation in the limit N >> 1

Measuring relaxation times

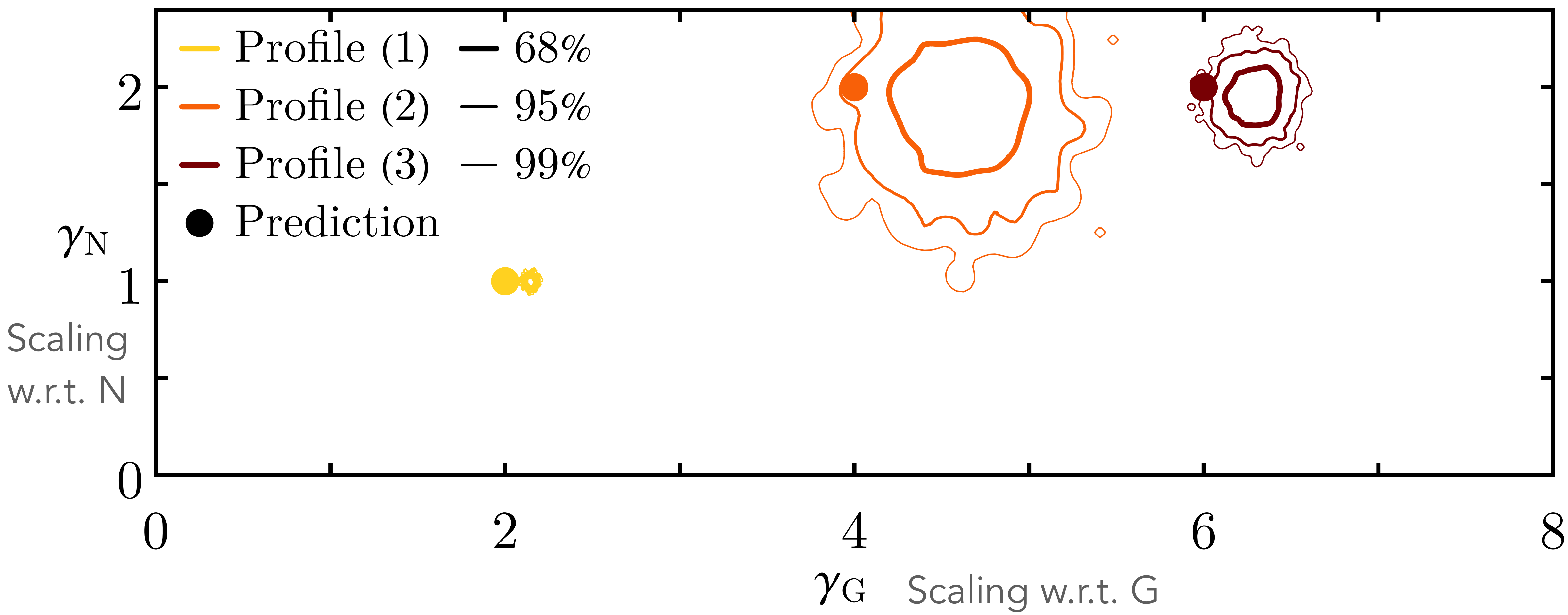
Relaxation time **w.r.t. N and G**

$$T_{\mathrm{relax}} \propto T_{\mathrm{dyn}}\, N^{\gamma_N} / G^{\gamma_G}$$

Frequency profiles

$$\begin{cases} (1): \Omega(J) = J; \\ (2): \Omega(J) = J\,|J|\,; \\ (3): \Omega(J) = J; \end{cases}$$

N-body measurements



Measuring relaxation times

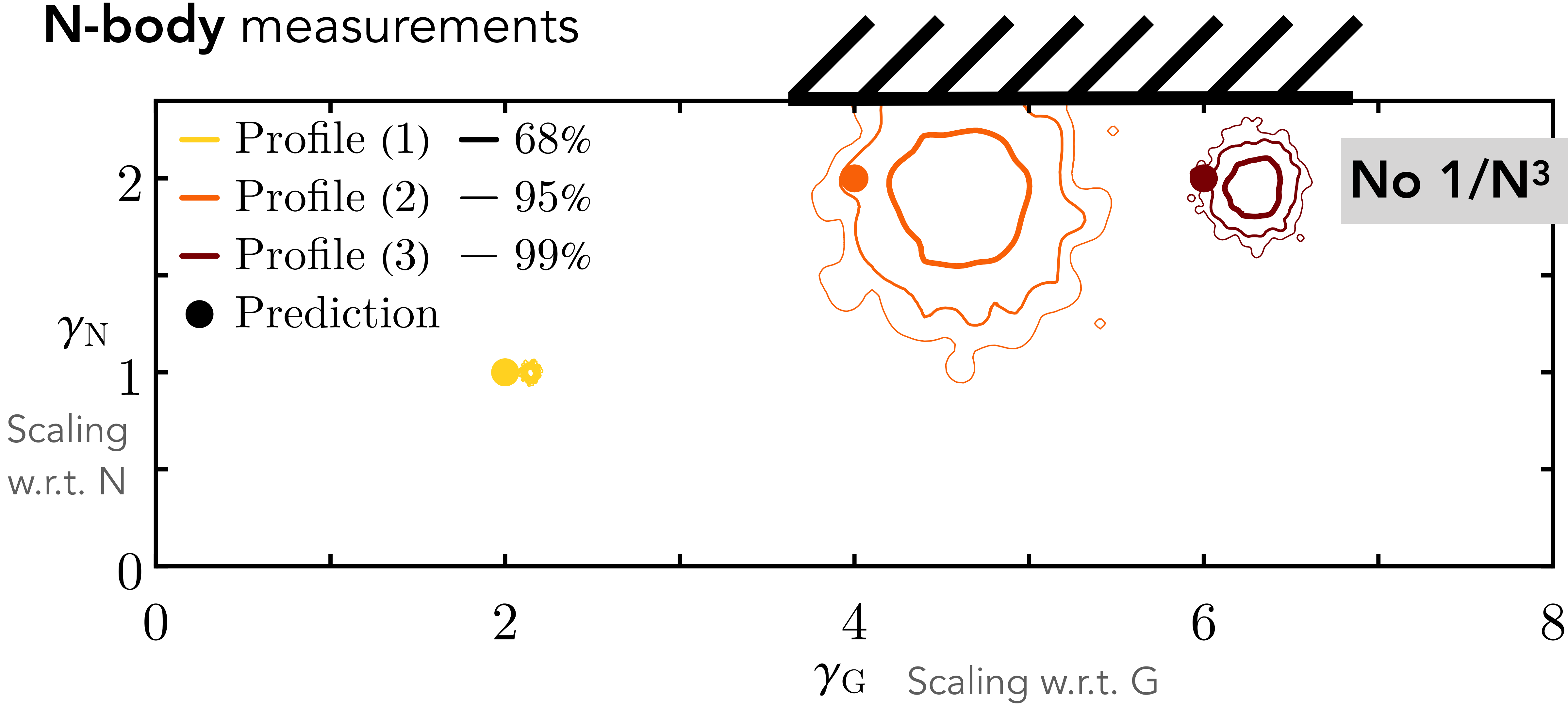
Relaxation time **w.r.t. N and G**

$$T_{\mathrm{relax}} \propto T_{\mathrm{dyn}}\, N^{\gamma_N} / G^{\gamma_G}$$

Frequency profiles

$$\begin{cases} (1): \Omega(J) = J; \\ (2): \Omega(J) = J|J|; \\ (3): \Omega(J) = J; \end{cases}$$

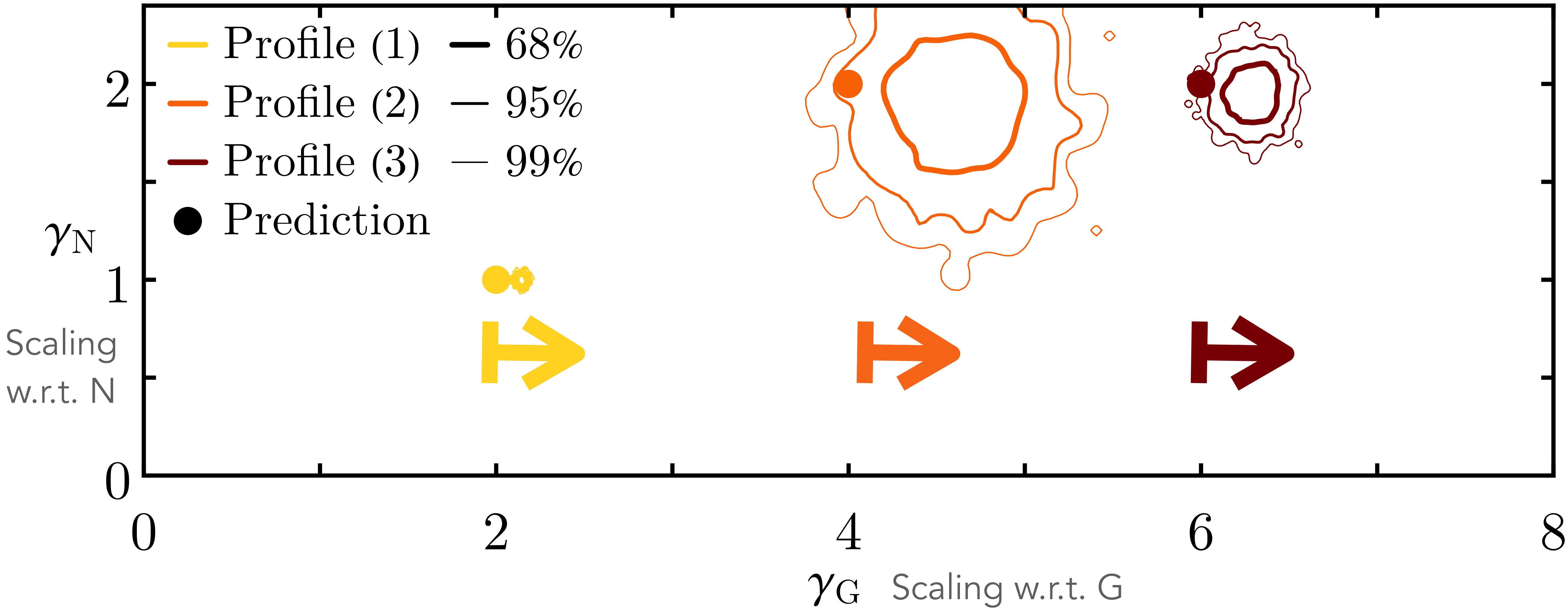
N-body measurements



No 1/N³ effects

Measuring relaxation times

An expected **bias**



Finite coupling amplitude

$$\frac{\partial F}{\partial t} \propto |U_{\mathbf{k}}^{\mathrm{d}}|^2 \propto \frac{1}{N}\left(\frac{G}{1-G}\right)^2 \underset{G>0}{\not\propto} \frac{G^2}{N}$$

Next steps

Collective effects

$$|\Lambda_{\mathbf{k}}(\mathbf{J})|^2 \to |\Lambda^{\mathrm{d}}_{\mathbf{k}}[F](\mathbf{J})|^2$$

Klimontovich approach

$$\frac{\partial F}{\partial t} \propto \left\langle \left[\delta F, \delta\Phi\right] \right\rangle$$

Large deviations

$$\mathbb{P}\left(\{F_N(t)\}_t = \{F(t)\}_t\right) = \,?$$

Marginal stability

$$\frac{1}{\epsilon_{\mathbf{k}}(\omega)} \to +\infty$$

Final equation

1D 1/**N²** inhomogeneous Landau equation

$$\frac{\partial F(J)}{\partial t} = 2\pi^3 m^2 \frac{\partial}{\partial J}\left[\sum_{k_1,k_2} \frac{k_1+k_2}{k_1^2(k_1+k_2)^2}\mathscr{P}\int \frac{\mathrm{d}J_1}{(\Omega[J]-\Omega[J_1])^4} \times \int \mathrm{d}J_2\, \mathcal{U}_{k_1k_2}(\mathbf{J})\,\delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}]\,\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}F_3(\mathbf{J})\right]$$

The DF appears three times

$$\mathbf{J} = \big(J, J_1, J_2\big); \quad F_3(\mathbf{J}) = F(J)\,F(J_1)\,F(J_2)$$

Three-body resonances

$$\mathbf{k} = \big(k_1+k_2, -k_1, -k_2\big); \quad \mathbf{\Omega} = \big(\Omega[J], \Omega[J_1], \Omega[J_2]\big)$$