# Linear response theory
# and
# damped modes of stellar clusters

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# Linear response theory

# Klimontovich equation

Describing one **realisation** in **phase space** $\mathbf{w} = (\mathbf{x}, \mathbf{v})$

Empirical DF

$$F_\mathrm{d}(\mathbf{w}, t) = \sum_{i=1}^{N} m\, \delta_\mathrm{D}(\mathbf{w} - \mathbf{w}_i(t))$$

3D gravitational systems

$$U_\mathrm{ext} = \frac{|\mathbf{v}|^2}{2}$$

$$U = -\frac{G}{|\mathbf{r} - \mathbf{r}'|}$$

Empirical Hamiltonian

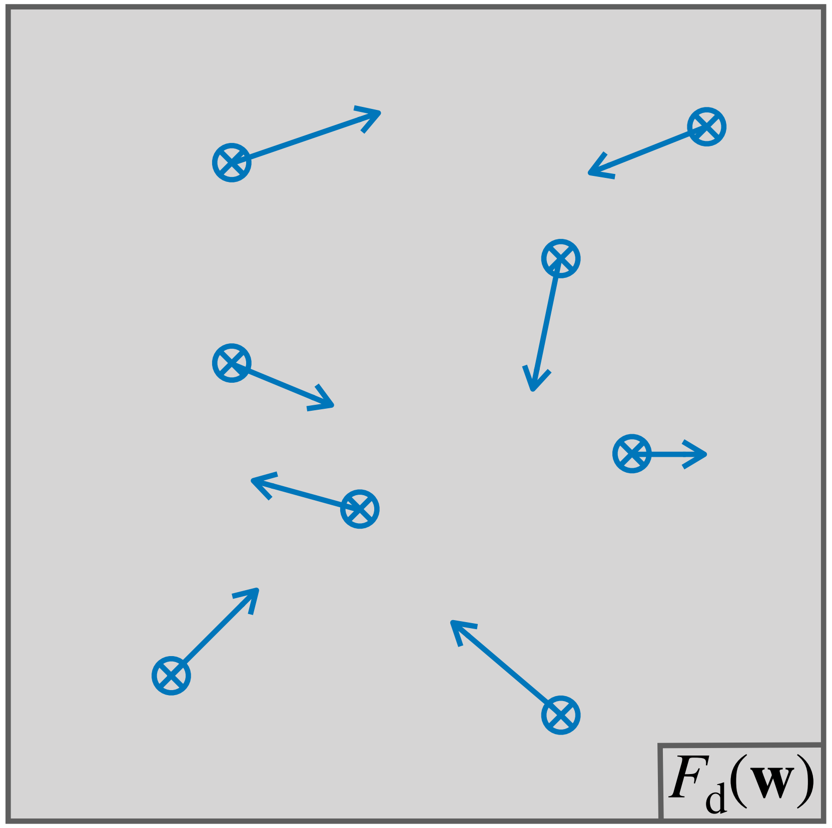
$$H_\mathrm{d}(\mathbf{w}, t) = U_\mathrm{ext}(\mathbf{w}) + \int \mathrm{d}\mathbf{w}'\, F_\mathrm{d}(\mathbf{w}', t)\, U(\mathbf{w}, \mathbf{w}')$$

**Continuity equation** in phase space

$$\frac{\partial F_\mathrm{d}}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot \left( F_\mathrm{d}\, \dot{\mathbf{w}} \right) = 0$$

Exact **Klimontovich** equation

$$\frac{\partial F_\mathrm{d}}{\partial t} + \left[ F_\mathrm{d}, H_\mathrm{d} \right] = 0$$

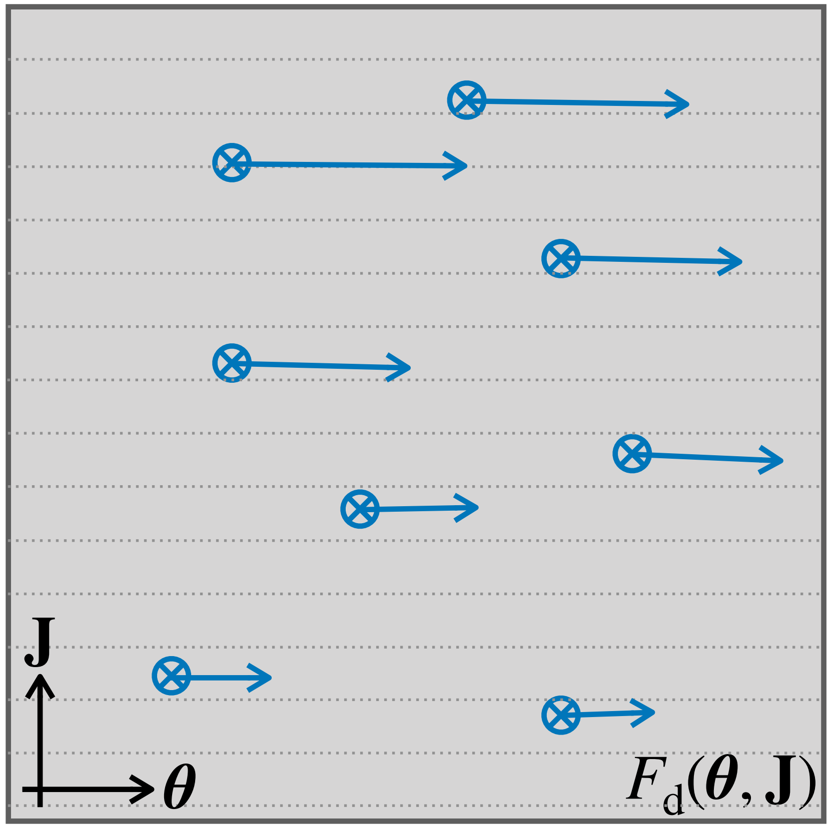



Phase space

# Solving Klimontovich

## **Perturbative** expansion

$$\begin{cases} F_\mathrm{d} = F_0 + \delta F \;\; \text{with} \;\; \langle \delta F \rangle = 0, \\ H_\mathrm{d} = H_0 + \delta H \;\; \text{with} \;\; \langle \delta H \rangle = 0. \end{cases}$$




Angle-Action space

## Adiabatic approximation

$$\begin{cases} F_0 = F_0(\mathbf{J}, t), \\ H_0 = H_0(\mathbf{J}, t)\,. \end{cases}$$

## Quasi-linear evolution equations

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta H\right] = 0$$

$$\frac{\partial F_0}{\partial t} = -\left\langle \left[\delta F, \delta H\right] \right\rangle$$

## Timescale separation

$$\begin{cases} T_{\delta F} \simeq T_\mathrm{dyn} \\ T_{F_0} \simeq \left(\sqrt{N}\right)^2 \times T_{\delta F} \end{cases}$$

# Dynamics of fluctuations

Fast evolution of **perturbations** (Linearised Klimontovich equation)

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta H\right] = 0$$

$\left[\delta F, H_0\right]$ Mean-field advection

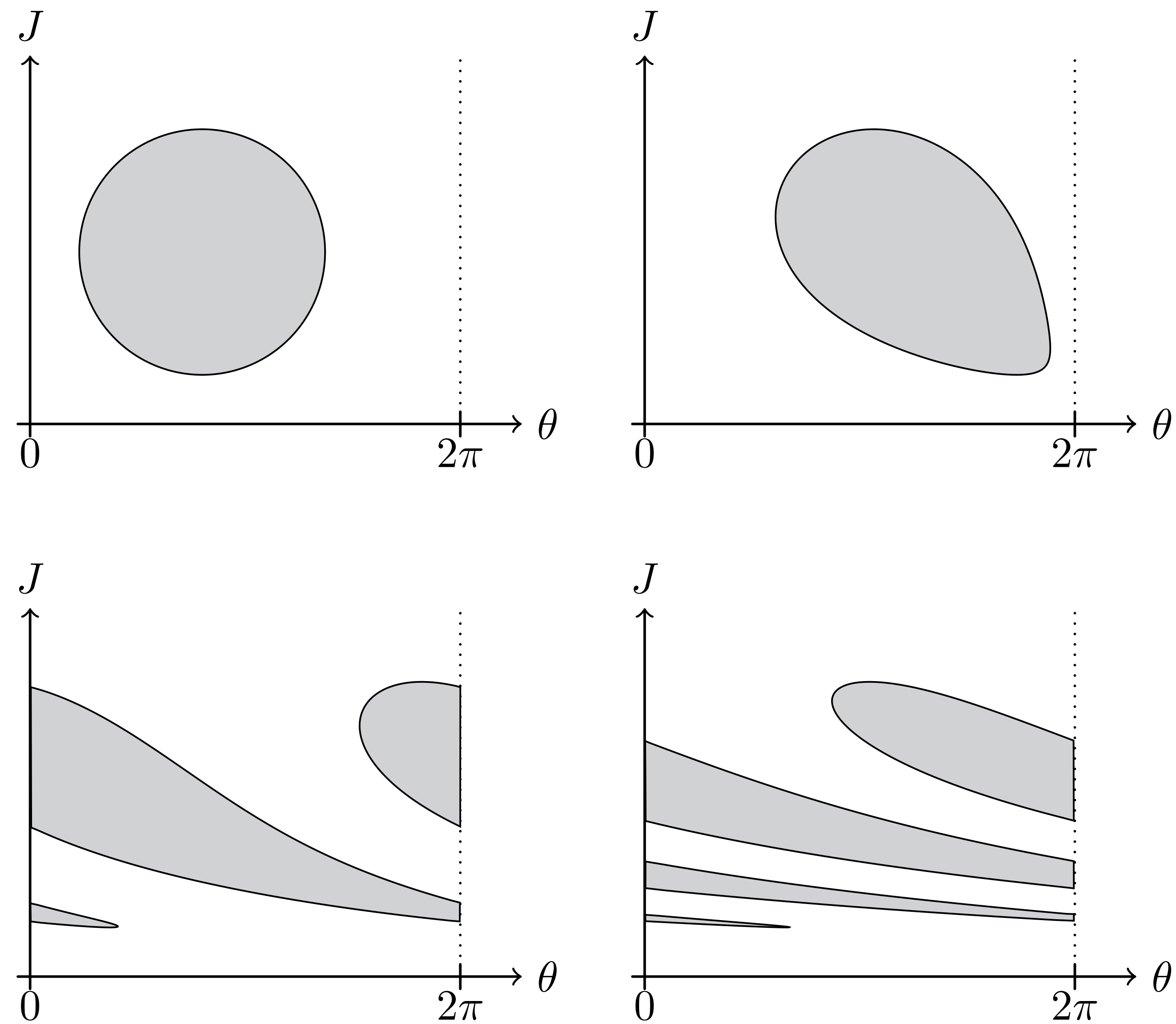
$\left[F_0, \delta H\right]$ Collective effects

**Self-consistent** amplification

$$\delta H = \delta H\left[\delta F\right]$$

**Timescale separation**

$$\begin{cases} F_0(\mathbf{J}) = \mathrm{cst} \\ H_0(\mathbf{J}) = \mathrm{cst} \end{cases}$$




**Phase Mixing**

# Solving for the fluctuations

Linear amplification

$$\delta \hat{F}_{\mathbf{k}}(\mathbf{J},\omega) = -\frac{\delta F_{\mathbf{k}}(\mathbf{J},0)}{\mathrm{i}(\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}))} - \frac{\mathbf{k}\cdot\partial F_0/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})}\,\delta \hat{H}_{\mathbf{k}}(\mathbf{J},\omega)$$

Bare noise

Self-consistent amplification

with the **self-consistency**

$$\delta H(\mathbf{w},t) = \int \mathrm{d}\mathbf{w}'\,\delta F(\mathbf{w}',t)\,U(\mathbf{w},\mathbf{w}') \qquad U = -\frac{G}{|\mathbf{r}-\mathbf{r}'|}$$

Generic form of a **Fredholm equation**

$$\big[\delta H(\mathbf{J})\big]_{\mathrm{dressed}} = \big[\delta H(\mathbf{J})\big]_{\mathrm{bare}} + \int \mathrm{d}\mathbf{J}'\,M(\mathbf{J},\mathbf{J}')\,\big[\delta H(\mathbf{J}')\big]_{\mathrm{dressed}}$$

Amplification kernel

**Dressing** of perturbations

$$\big[\delta H(\omega)\big]_{\mathrm{dressed}} \simeq \frac{[\delta H(\omega)]_{\mathrm{bare}}}{1 - M(\omega)} = \frac{[\delta H(\omega)]_{\mathrm{bare}}}{|\,\varepsilon(\omega)\,|}$$

# Solving for the fluctuations

Linear amplification

$$\delta\hat{F}_{\mathbf{k}}(\mathbf{J},\omega) = -\frac{\delta F_{\mathbf{k}}(\mathbf{J},0)}{\mathrm{i}(\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))} - \frac{\mathbf{k}\cdot\partial F_0/\partial\mathbf{J}}{\omega - \mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J})}\,\delta\hat{H}_{\mathbf{k}}(\mathbf{J},\omega)$$

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Amplification kernel

**Plasma dielectric function**

$$\varepsilon_{\mathbf{k}}(\omega) = 1 + \frac{1}{k^2\lambda_{\mathrm{D}}^2}\int \mathrm{d}\mathbf{v}\,\frac{\mathbf{k}\cdot\partial F/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v} - \omega}$$

# Gravitational response matrix

$$\boldsymbol{\varepsilon}_{pq}(\omega) = \mathbf{I} - \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{J}}{\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \omega} \, \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \, \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Some properties

$\sum_{\mathbf{k}}$ *Sum over resonances*

$\int \mathrm{d}\mathbf{J}$ *Scan over orbital space*

$\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \omega$ *Resonant amplification*

$\psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \, \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$ *Long-range interaction*

Mode

$$\det\left[\boldsymbol{\varepsilon}(\omega)\right] = 0$$

Type of modes

$$\begin{cases} \mathrm{Im}[\omega] > 0 & \text{Unstable} \\ \mathrm{Im}[\omega] = 0 & \text{Neutral} \\ \mathrm{Im}[\omega] < 0 & \text{Damped} \end{cases}$$

| Plasmas | | Galaxies |
|---|---|---|

## Orbital coordinates

$(\mathbf{x}, \mathbf{v})$ | $(\boldsymbol{\theta}, \mathbf{J})$

## Basis decomposition

$$U(\mathbf{x}, \mathbf{x}') \propto \int \frac{\mathrm{d}\mathbf{k}}{\mathbf{k}^2}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}$$

$$U(\mathbf{w}, \mathbf{w}') = -\sum_p \psi^{(p)}(\mathbf{w})\, \psi^{(p)*}(\mathbf{w}')$$

## Dielectric function

$$1 + \frac{1}{\mathbf{k}^2} \int \mathrm{d}\mathbf{v} \frac{\mathbf{k}\cdot \partial F/\partial \mathbf{v}}{\mathbf{k}\cdot\mathbf{v} - \omega}$$

$$\delta_{pq} - \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \frac{\mathbf{k}\cdot \partial F/\partial \mathbf{J}}{\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \omega} \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

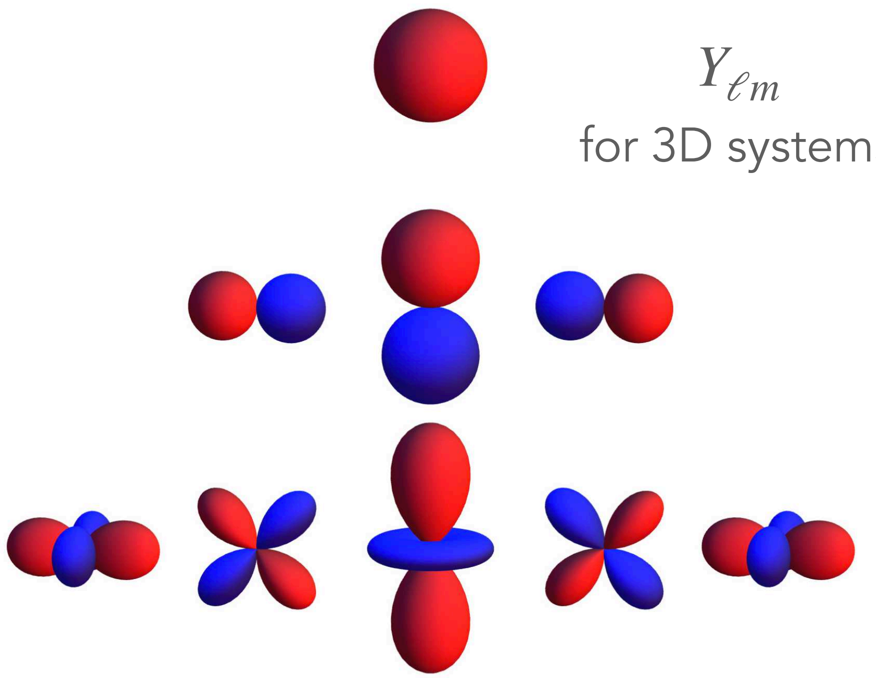
## Resonance condition

$$\delta_{\mathrm{D}}\big(\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')\big)$$

$$\delta_{\mathrm{D}}\big(\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}')\big)$$

# How to compute the dispersion function?

**Basis method** $\left(\psi^{(p)}(\mathbf{w}), \rho^{(p)}(\mathbf{w})\right)$




$$\begin{cases} \psi^{(p)}(\mathbf{w}) = \int \mathrm{d}\mathbf{w}' \, U(\mathbf{w}, \mathbf{w}') \, \rho^{(p)}(\mathbf{w}'), \\ \int \mathrm{d}\mathbf{w} \, \psi^{(p)}(\mathbf{w}) \, \rho^{(q)*}(\mathbf{w}) = -\,\delta_{pq} \, . \end{cases}$$

``**Separable**'' pairwise interaction

$$U(\mathbf{w}, \mathbf{w}') = -\sum_p \psi^{(p)}(\mathbf{w}) \, \psi^{(p)*}(\mathbf{w}')$$

**Plasmas**

$$U(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}$$

$$\simeq \int \frac{\mathrm{d}\mathbf{k}}{|\mathbf{k}|^2} \, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}$$

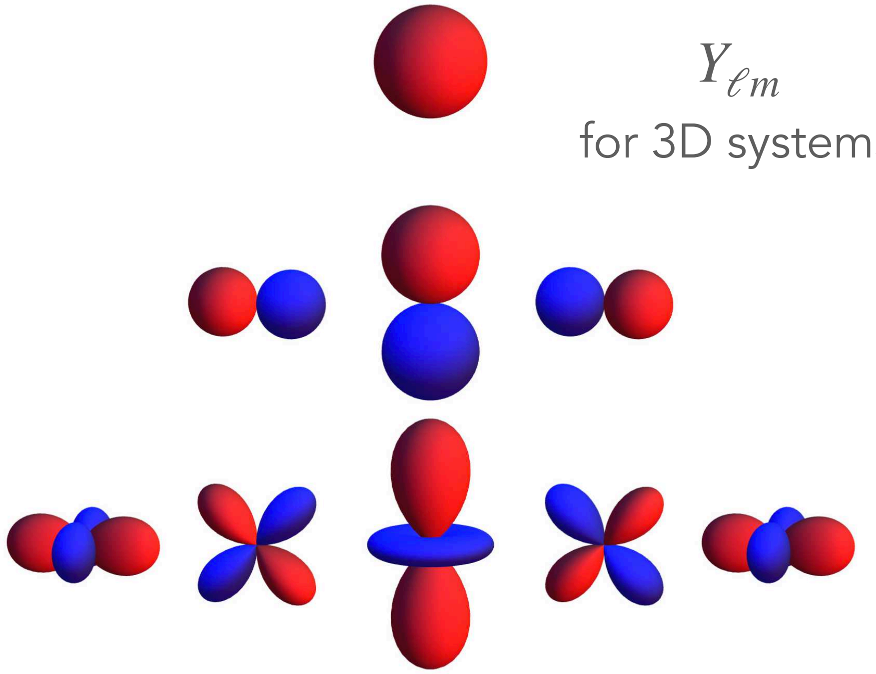
**Galaxies**

$$\Delta\Phi = 4\pi G\rho$$

Poisson equation

**Basis method** $\left(\psi^{(p)}(\mathbf{w}), \rho^{(p)}(\mathbf{w})\right)$

$$\begin{cases} \psi^{(p)}(\mathbf{w}) = \displaystyle\int \mathrm{d}\mathbf{w}'\, U(\mathbf{w}, \mathbf{w}')\, \rho^{(p)}(\mathbf{w}'), \\ \displaystyle\int \mathrm{d}\mathbf{w}\, \psi^{(p)}(\mathbf{w})\, \rho^{(q)*}(\mathbf{w}) = -\,\delta_{pq}\,. \end{cases}$$



$Y_{\ell m}$
for 3D systems

**Newtonian** interaction

$$U(\mathbf{r}, \mathbf{r}') = -\frac{G}{|\mathbf{r} - \mathbf{r}'|}$$

*Scale invariance*

$$= -\int \frac{\mathrm{d}\mathbf{k}}{\mathbf{k}^2}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}$$

*Translation invariance*

$$= -\sum_{\ell,m} Y_{\ell m}(\hat{\mathbf{r}})\, Y_{\ell m}(\hat{\mathbf{r}}')\, \frac{\mathrm{Min}[r, r']^{\ell}}{\mathrm{Max}[r, r']^{\ell+1}}$$

*Rotation invariance*

# Biorthogonal basis

What matters is the **mean potential**

$$\begin{cases} \rho_{\ell=0,n=1}(r) = \rho_0(r), \\ \rho_{\ell=1,n=1}(r) = \mathrm{d}\rho_0/\mathrm{d}r, \\ \ldots \end{cases}$$

*cf. Self-consistent field simulations*

What matters are the **perturbations**

$$\delta\rho(\mathbf{r},t) = \sum_p A_p(t)\,\rho^{(p)}(\mathbf{r})$$

*cf. Linear response in time domain*

What matters is the **pairwise interaction**

$$U(\mathbf{r},\mathbf{r}') = -\sum_p \psi^{(p)}(\mathbf{r})\,\psi^{(p)*}(\mathbf{r}')$$

*cf. Kinetic theory*

**How to chose the basis?**

# Self-consistent amplification

Linear response

$$\big[\delta H(\mathbf{J})\big]_{\mathrm{dressed}} = \big[\delta H(\mathbf{J})\big]_{\mathrm{bare}} + \int \mathrm{d}\mathbf{J}'\, M(\mathbf{J},\mathbf{J}')\, \big[\delta H(\mathbf{J}')\big]_{\mathrm{dressed}}$$

Amplification kernel

In terms of **coupling coefficients**

$$\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega) = \psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}') + (2\pi)^d \sum_{\mathbf{k}''} \int \mathrm{d}\mathbf{J}'' \frac{\mathbf{k}''\cdot\partial F/\partial\mathbf{J}''}{\mathbf{k}''\cdot\mathbf{\Omega}(\mathbf{J}'')-\omega}\, \psi_{\mathbf{k}\mathbf{k}''}(\mathbf{J},\mathbf{J}'')\, \psi^{\mathrm{d}}_{\mathbf{k}''\mathbf{k}}(\mathbf{J}'',\mathbf{J}',\omega)$$

$\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')$ **Bare** coefficient, **Landau**

$\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega)$ **Dressed** coefficient, **Balescu-Lenard**

**Can one compute the dressed coefficients without any basis?**

# Gravitational response matrix

$$\boldsymbol{\varepsilon}_{pq}(\omega) = \mathbf{I} - \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k} \cdot \partial F/\partial \mathbf{J}}{\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega} \, \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \, \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Some properties

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Mode

$$\det\left[\boldsymbol{\varepsilon}(\omega)\right] = 0$$

Type of modes

$$\begin{cases} \mathrm{Im}[\omega] > 0 & \text{Unstable} \\ \mathrm{Im}[\omega] = 0 & \text{Neutral} \\ \mathrm{Im}[\omega] < 0 & \text{Damped} \end{cases}$$

# Landau's prescription

$$\int_{-\infty \atop \mathscr{L}}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} & \text{if } \; \mathrm{Im}[\omega] > 0 \\ \displaystyle\mathscr{P}\int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} + \mathrm{i}\pi G(\omega) & \text{if } \; \mathrm{Im}[\omega] = 0 \\ \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} + 2\mathrm{i}\pi G(\omega) & \text{if } \; \mathrm{Im}[\omega] < 0 \end{cases}$$

***Unstable***, *e.g., ROI*

***Neutral***, *e.g., BL*

***Damped***, *e.g., sloshing*

Some remarks

``**Causality** breaking''

$$+\,\mathrm{Im}[\omega] \quad \text{vs} \quad -\,\mathrm{Im}[\omega]$$

``**Aligned**'' resonant denominator

$$\frac{1}{u-\omega} \quad \text{vs} \quad \frac{1}{f(u)-\omega}$$

**Analytic** integrand

$$G(\omega) \quad \text{for} \quad \omega \in \mathbb{C}$$

**Infinite** frequency support

$$\int_{-\infty}^{+\infty} \mathrm{d}u \quad \text{vs} \quad \int_{-1}^{1} \mathrm{d}u$$

# Landau's prescription

$$\int_{-\infty \atop \mathscr{L}}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} & \text{if } \mathrm{Im}[\omega] > 0 \\ \displaystyle\mathscr{P}\int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} + \mathrm{i}\pi G(\omega) & \text{if } \mathrm{Im}[\omega] = 0 \\ \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{G(u)}{u-\omega} + 2\mathrm{i}\pi G(\omega) & \text{if } \mathrm{Im}[\omega] < 0 \end{cases}$$

***Unstable***, *e.g., ROI*

***Neutral***, *e.g., BL*

***Damped***, *e.g., sloshing*

**Plasmas**

$$\varepsilon_{\mathbf{k}}(\omega) = 1 + \frac{1}{k^2\lambda_{\mathrm{D}}^2} \int \mathrm{d}\mathbf{v} \, \frac{\mathbf{k}\cdot\partial F/\partial \mathbf{v}}{\mathbf{k}\cdot\mathbf{v} - \omega}$$

Resonant denominator is aligned $u = \mathbf{k}\cdot\mathbf{v}$

Integrand is typically **analytic** $F(v) \propto \mathrm{e}^{-v^2/2}$

Frequency support is typically **infinite** $\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}v$

\`\`Vanilla'' Maxwellian case `Z[zeta_] := I Sqrt[Pi] Exp[-zeta^2](1 + I Erfi[zeta])`

# Aligning the denominator

One can equivalently label **orbits** with their **frequencies**

$$M(\omega) = \int \mathrm{d}\mathbf{J} \, \frac{G(\mathbf{J})}{\mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J}) - \omega}$$

*To* ***resonant frequency***

$$u \propto \mathbf{k} \cdot \mathbf{\Omega}(\mathbf{J})$$

$$M(\omega) = \int_{-1}^{1}{}_{\mathscr{L}} \mathrm{d}u \, \frac{G(u)}{u - \omega}$$

# Analytic continuation

Initial expression

$$M(\omega) = \int_{-1}^{1} \mathrm{d}u \, \frac{G(u)}{u - \omega} \quad (\mathscr{L})$$

On $[-1, 1]$ with unit weight: **Legendre projection**

$$G(u) = \sum_k a_k P_k(u)$$

*Polynomial, therefore analytic*

Hence the **separable** writing

$$M(\omega) = \sum_k a_k D_k(\omega)$$

with

$$D_k(\omega) = \int_{-1}^{1} \mathrm{d}u \, \frac{P_k(u)}{u - \omega} \quad (\mathscr{L})$$

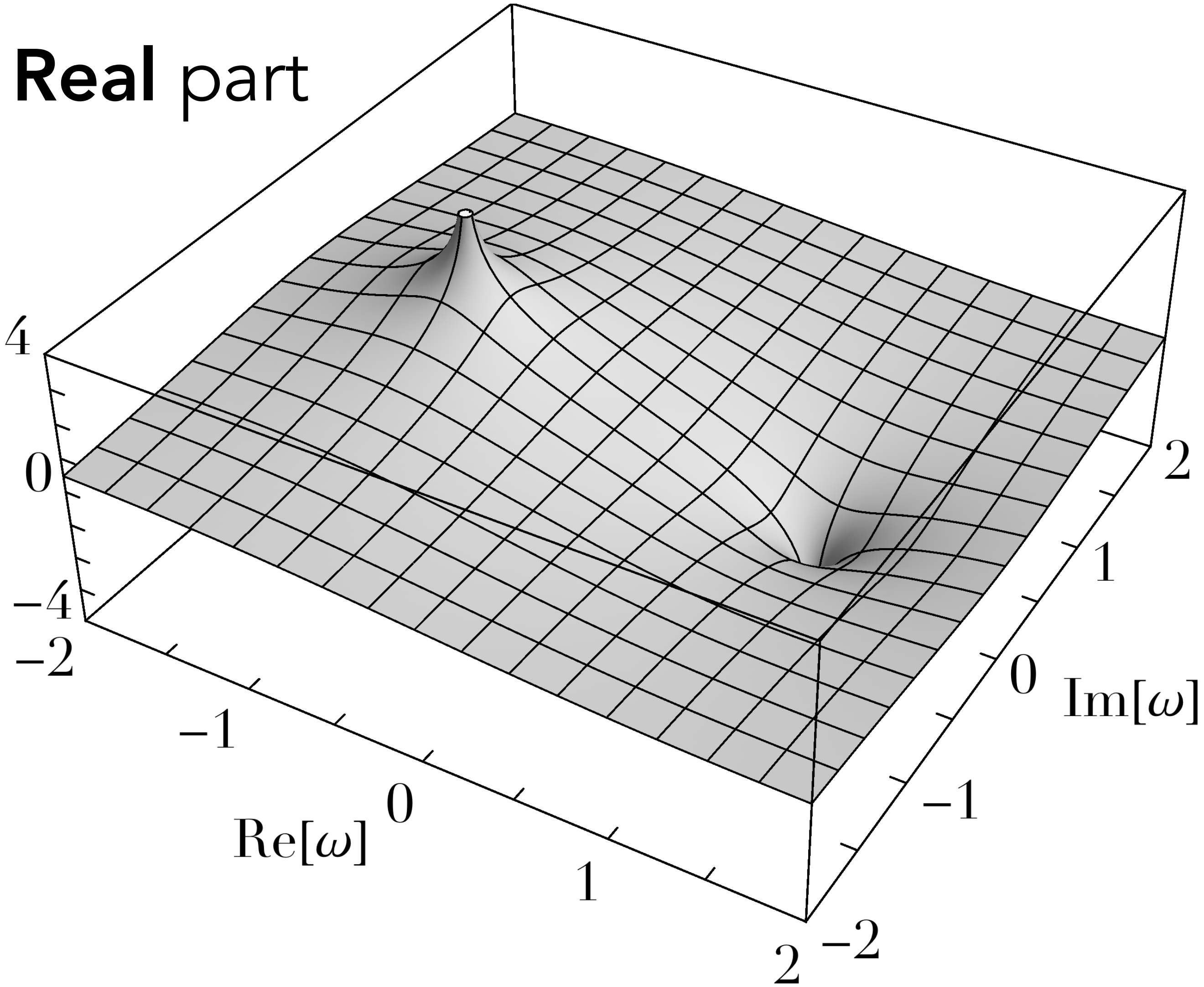
$\{F(\mathbf{J}), \mathbf{\Omega}(\mathbf{J}), \psi^{(p)}(\mathbf{r})\}$ → $a_k$

# The resonant integral

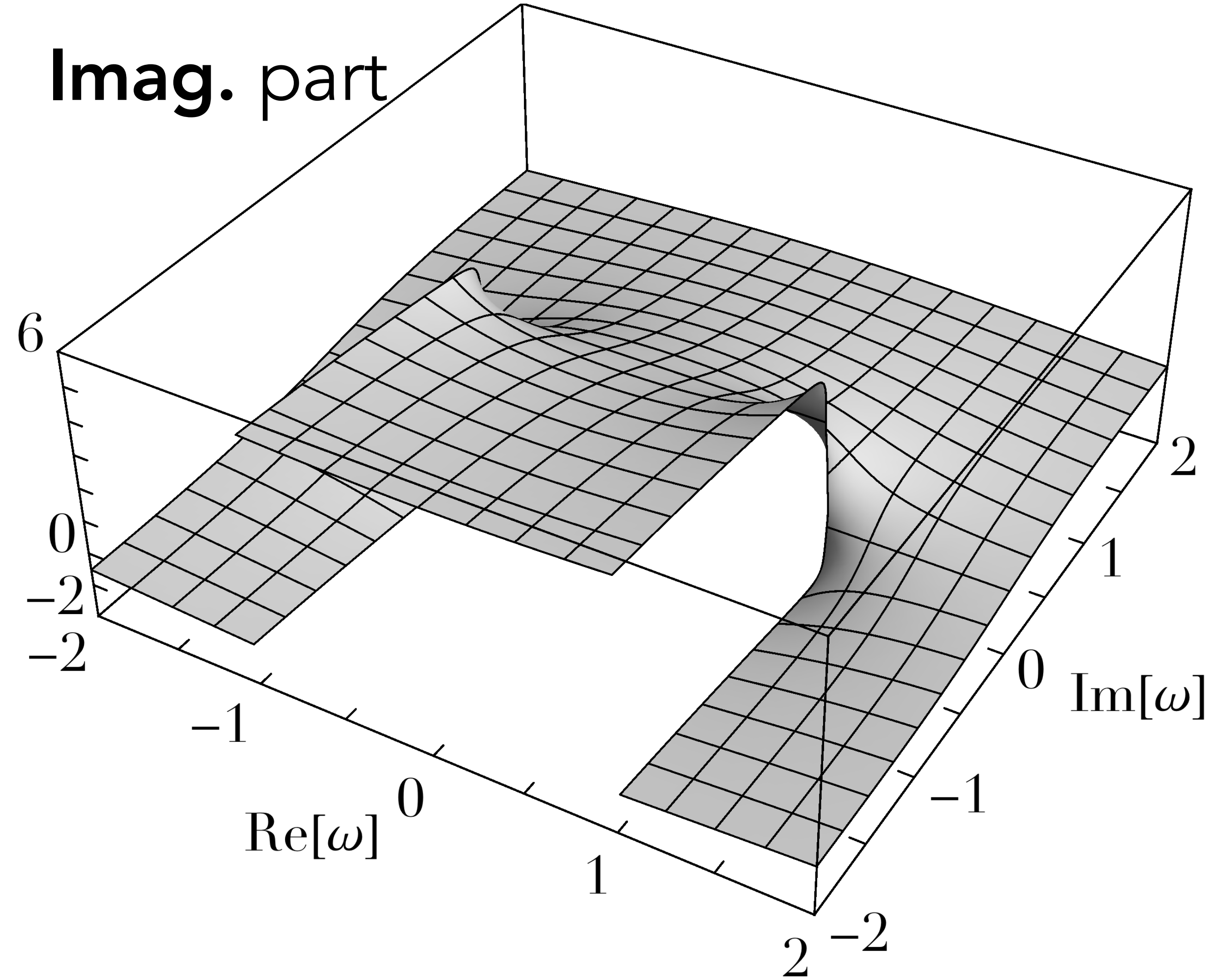
**Finite** frequency-domain

$$D_0(\omega) = \int_{-1}^{1}_{\mathscr{L}} \mathrm{d}u \, \frac{1}{u - \omega}$$

**Real** part




**Imag.** part



# Only one difficult integral

We know the one integral

$$D_0(\omega) = \int_{-1}^{1} \mathrm{d}u \, \frac{1}{u - \omega}$$
$$\mathscr{L}$$

``Pain de sucre''

$$D_1(\omega) = \int_{-1}^{1} \mathrm{d}u \, \frac{u}{u - \omega} = \int_{-1}^{1} \mathrm{d}u \, \frac{u - (\omega - \omega)}{u - \omega}$$
$$\mathscr{L} \qquad \mathscr{L}$$
$$= 2 + \omega \, D_0(\omega)$$

**Legendre recurrence** gives $P_{k+2}(\omega) = \mathrm{Linear}\big[P_k(\omega), P_{k+1}(\omega)\big]$

Hence, we know all

$$D_k(\omega) = \int_{-1}^{1} \mathrm{d}u \, \frac{P_k(u)}{u - \omega}$$
$$\mathscr{L}$$

# Ready to compute

Generic expression

$$M_{pq}(\omega) = \sum_{\mathbf{k}} \sum_{k} a_k[p, q, \mathbf{k}] \, D_k(\varpi_{\mathbf{k}})$$

**Projection** to get $\{a_k\}$

$$\mathcal{O}\left[K \times N_{\mathrm{radial}}^2 \times k_1^{\mathrm{max}} \times \ell_{\mathrm{max}} \times K_u \times K_v\right]$$

**Evaluation** to get $\mathbf{M}(\omega)$

$$\mathcal{O}\left[N_{\mathrm{radial}}^2 \times k_1^{\mathrm{max}} \times \ell_{\mathrm{max}} \times K_u\right]$$

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$K$ *Sampling of the orbit-average*

$N_{\mathrm{radial}}$ *Number of basis elements*

$k_1^{\mathrm{max}}$ *Number of radial resonances*

$\ell_{\mathrm{max}}$ *Considered harmonics*

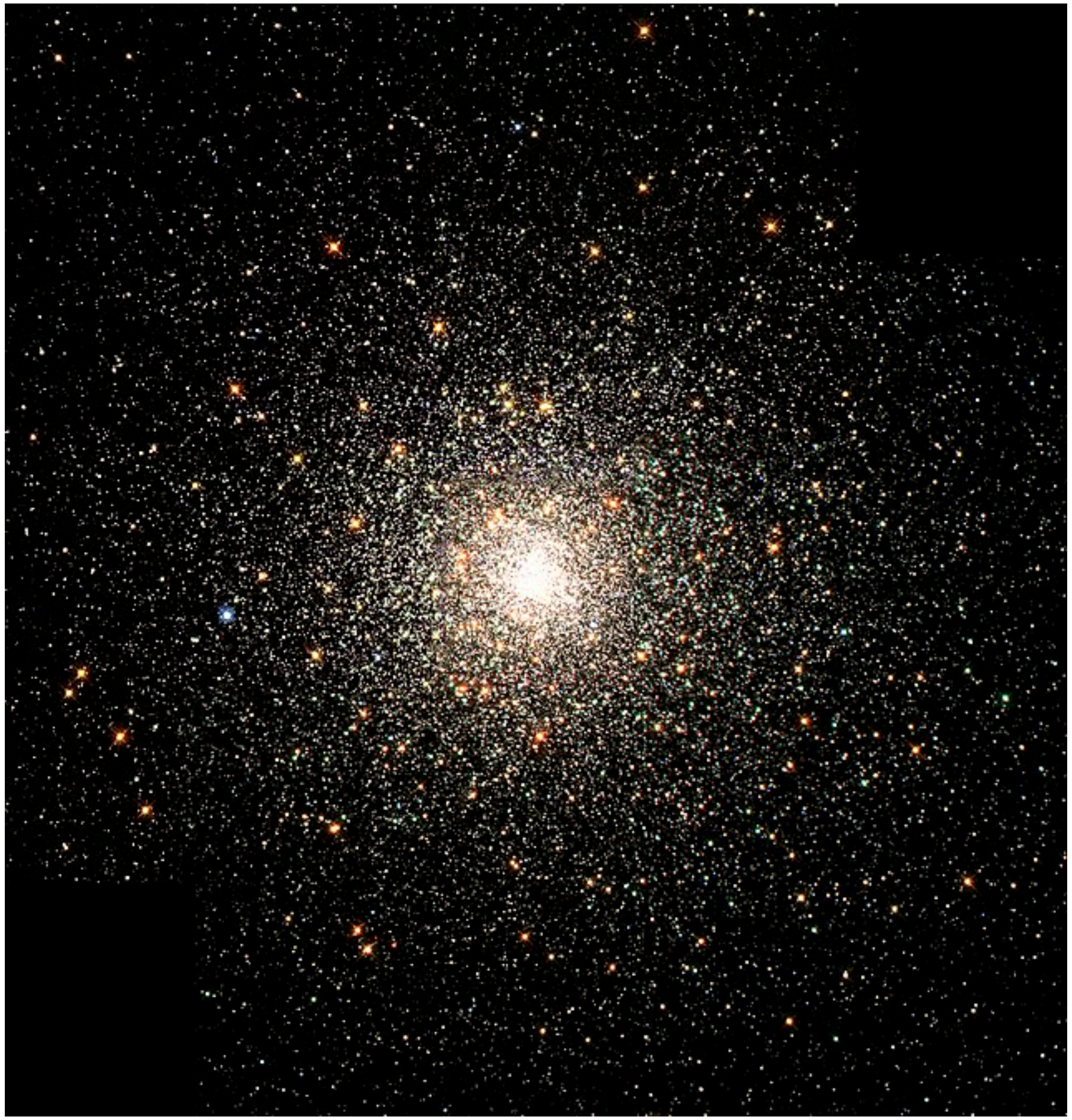
$K_u$ *Number of Legendre functions*

$K_v$ *Number of sampling 2nd dim.*

# Damped modes
# in globular clusters

# Globular clusters

**Dense**, spherical stellar systems



Radii ~ a few parsecs

Contains $N \sim 10^5$ stars

Very old $\sim 10^{10}$ yr

Crossing time $\sim 10^5$ yr

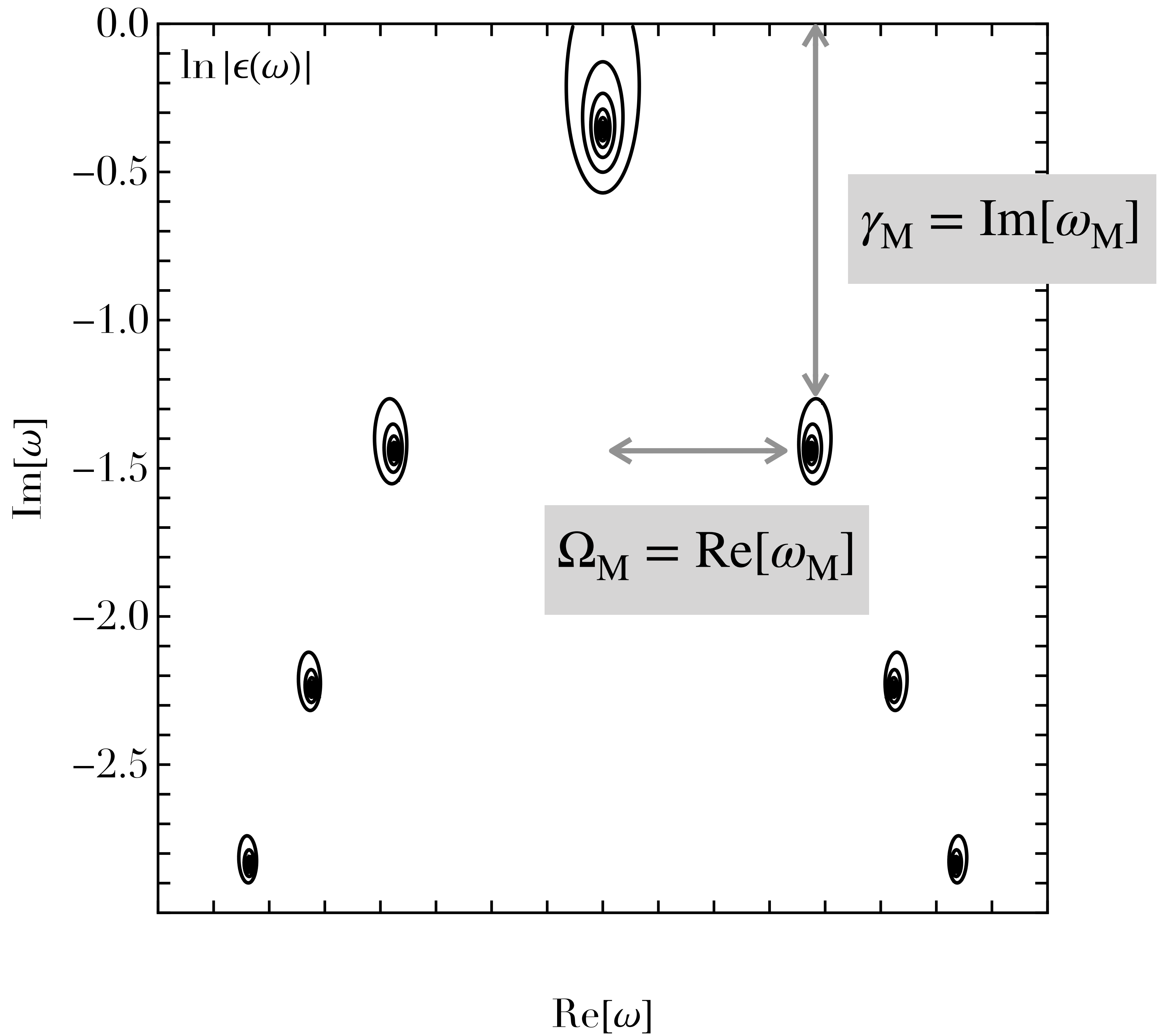
Relaxation time $\sim 10^{10}$ yr

Expected to be **linearly stable**

No maximum entropy, i.e. no Maxwellian

# Dispersion function

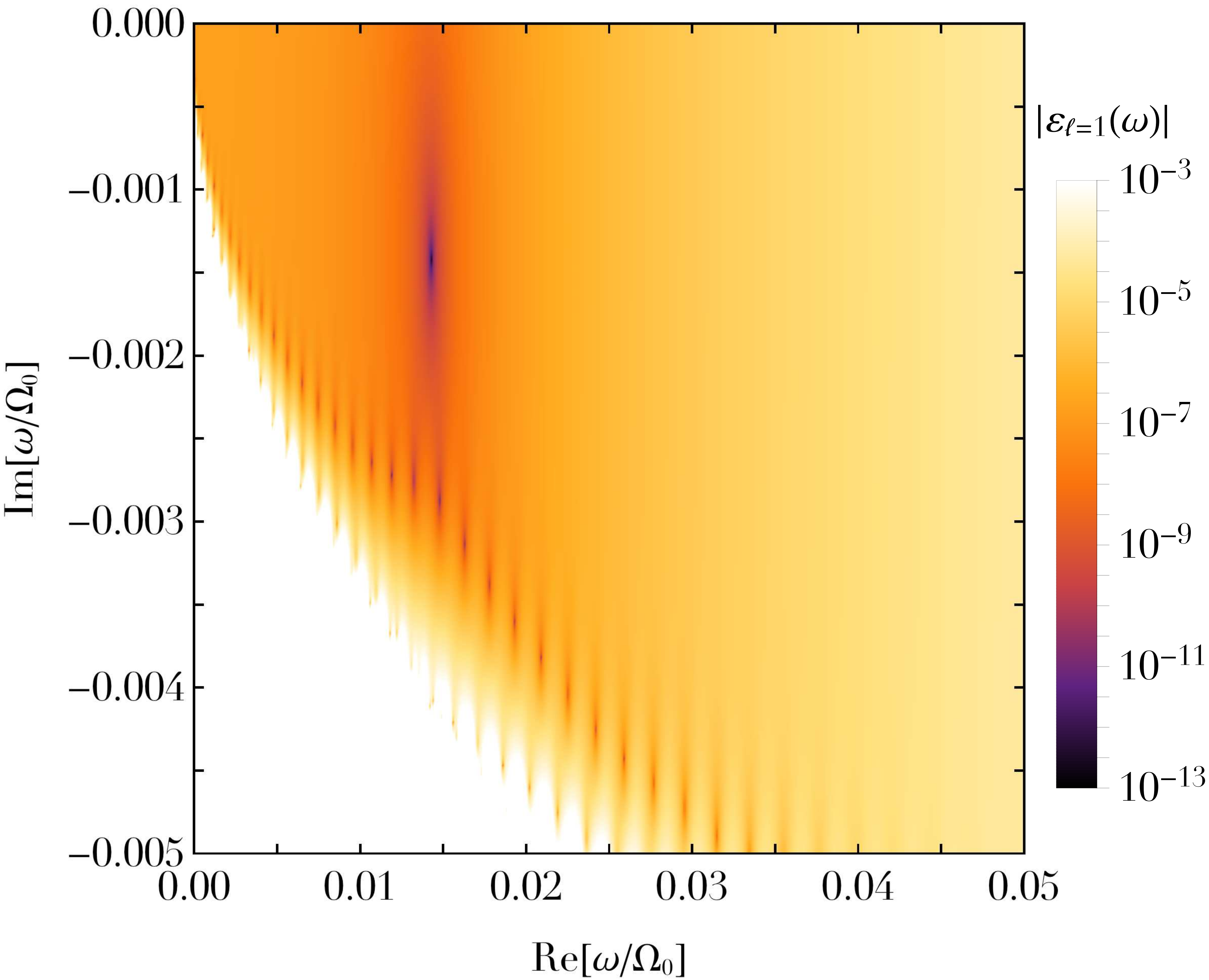
(Landau) damped modes in a (periodic) **plasma**



# Dispersion function

Isotropic isochrone cluster

$\det\left[\boldsymbol{\varepsilon}_{\ell}(\omega)\right]$




$\ell = 1$ **damped** mode

$$\omega_{\mathrm{M}}/\Omega_0 = 0.0143 - 0.00142\,\mathrm{i}$$

**Slow** mode

$$\mathrm{Re}[\omega_{\mathrm{M}}]/\Omega_0 \ll 1$$

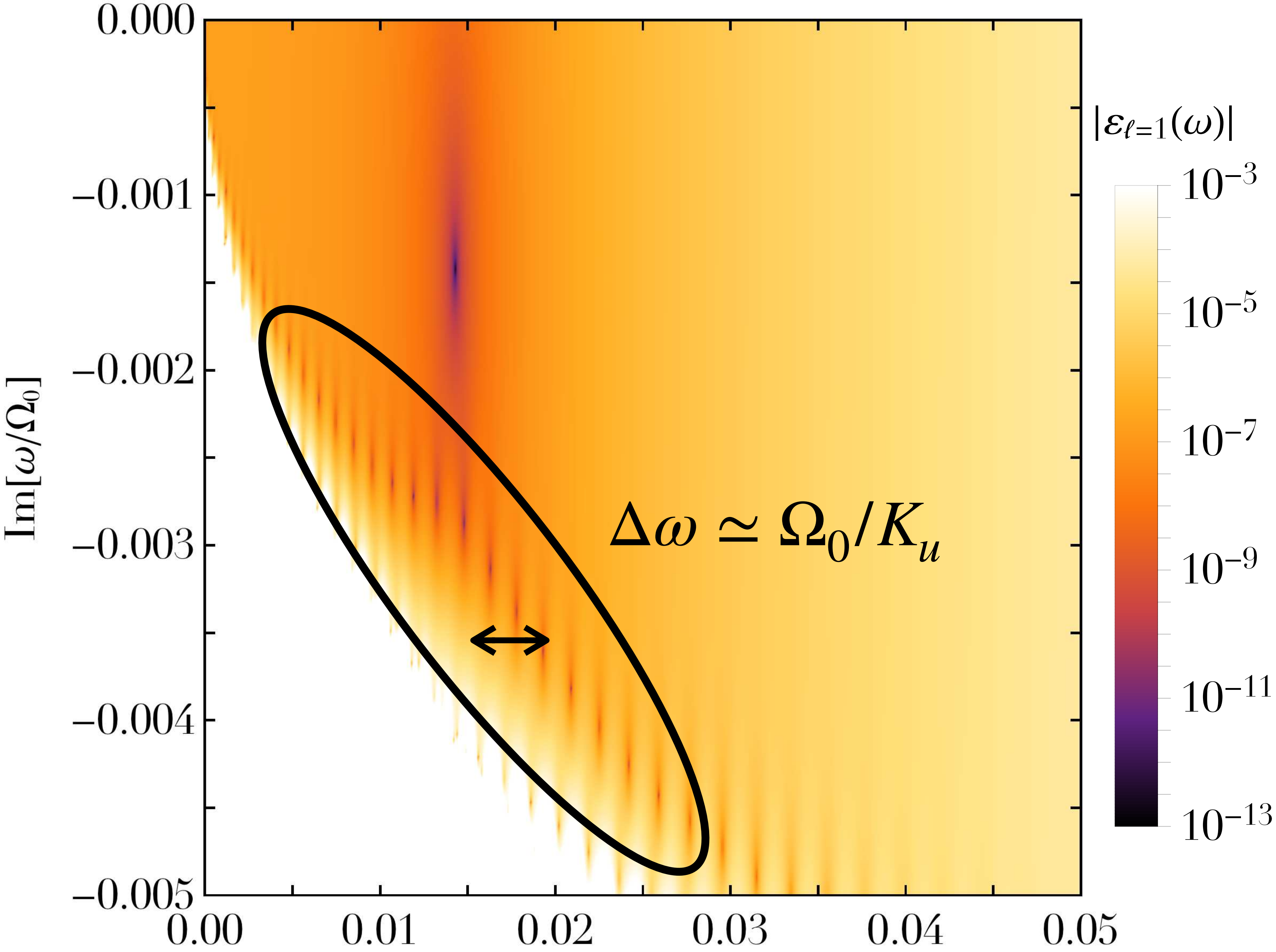
**Weakly** damped

$$\mathrm{Im}[\omega_{\mathrm{M}}]/\mathrm{Re}[\omega_{\mathrm{M}}] \ll 1$$

# Dispersion function

Isotropic isochrone cluster

$\det[\boldsymbol{\varepsilon}_{\ell}(\omega)]$

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**Slow** mode

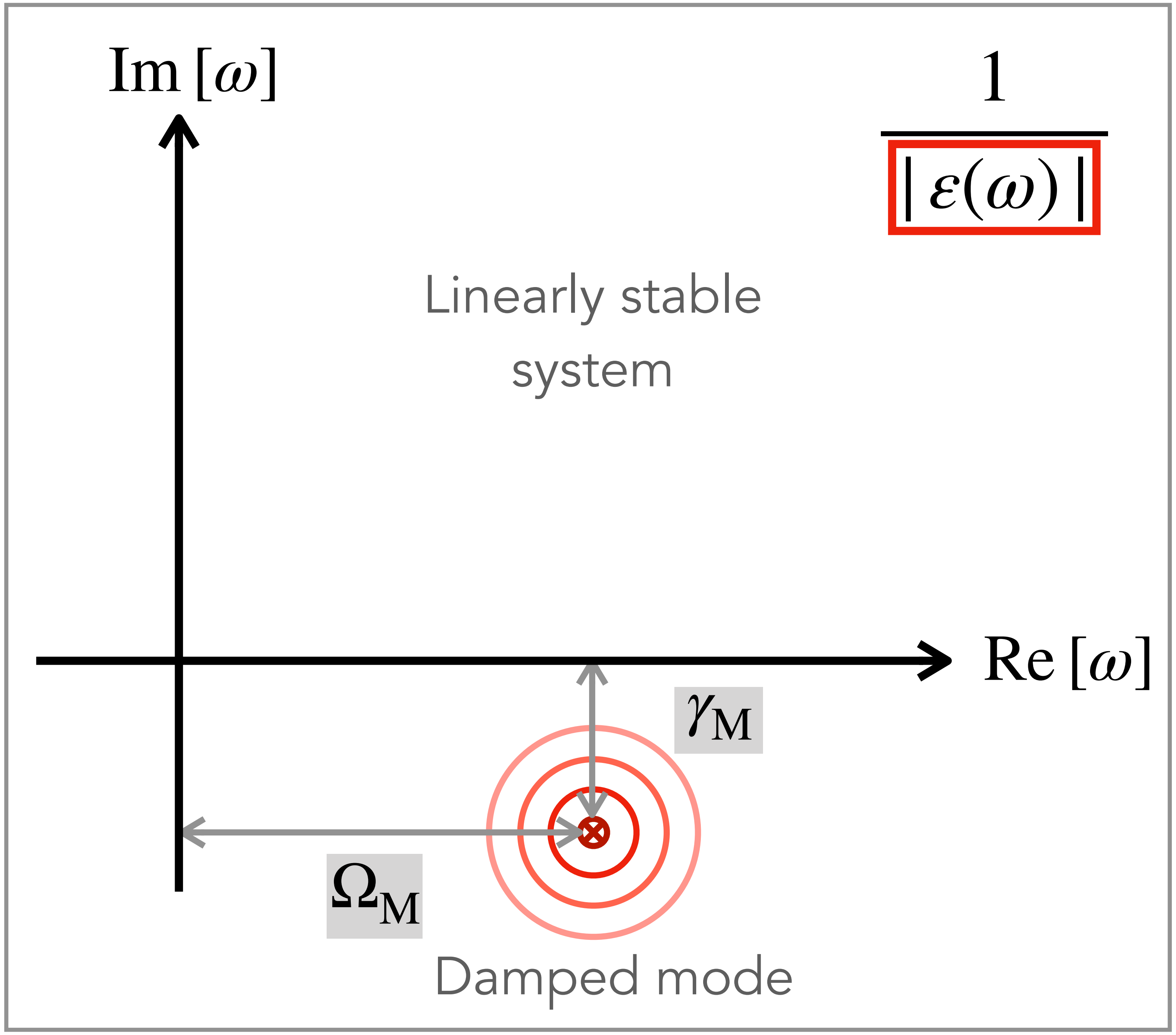
$$\mathrm{Re}[\omega_{\mathrm{M}}]/\Omega_0 \ll 1$$

**Weakly** damped

$$\mathrm{Im}[\omega_{\mathrm{M}}]/\mathrm{Re}[\omega_{\mathrm{M}}] \ll 1$$

**How to reduce the spurious oscillations stemming from Legendre?**

# Amplification




## Susceptibility

$$\frac{1}{|\varepsilon(\Omega_{\mathrm{M}})|} \gg 1$$

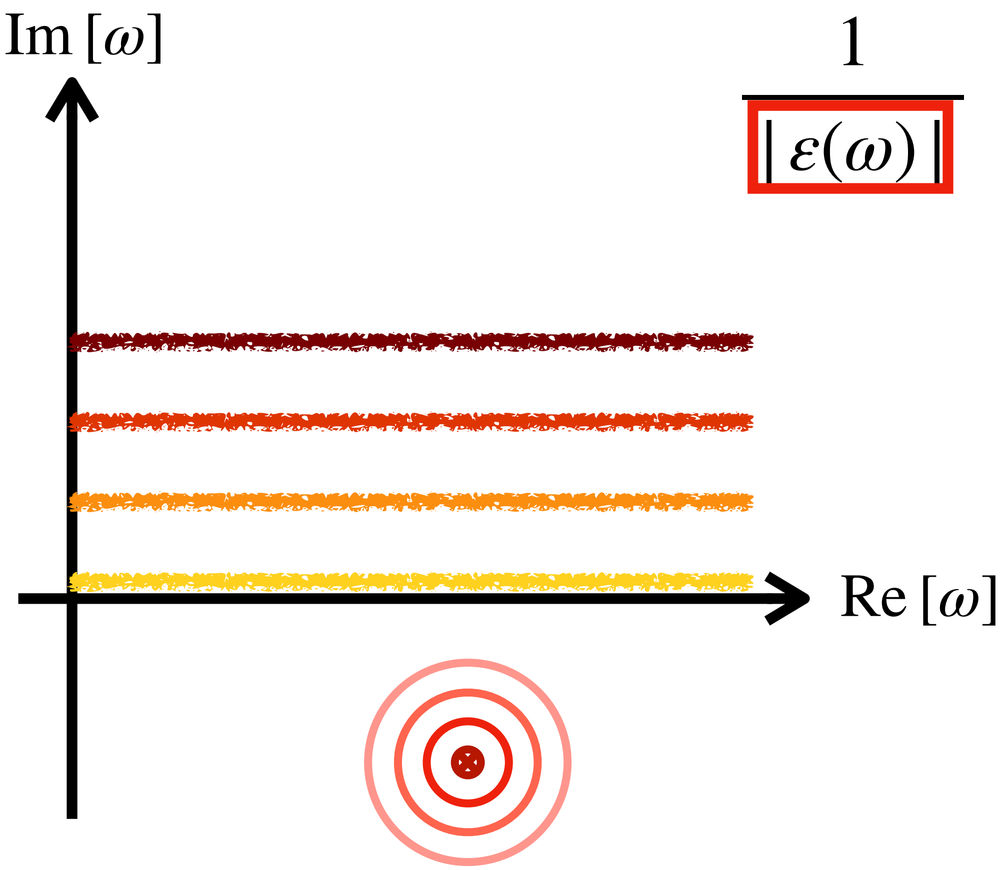
## Thermalisation

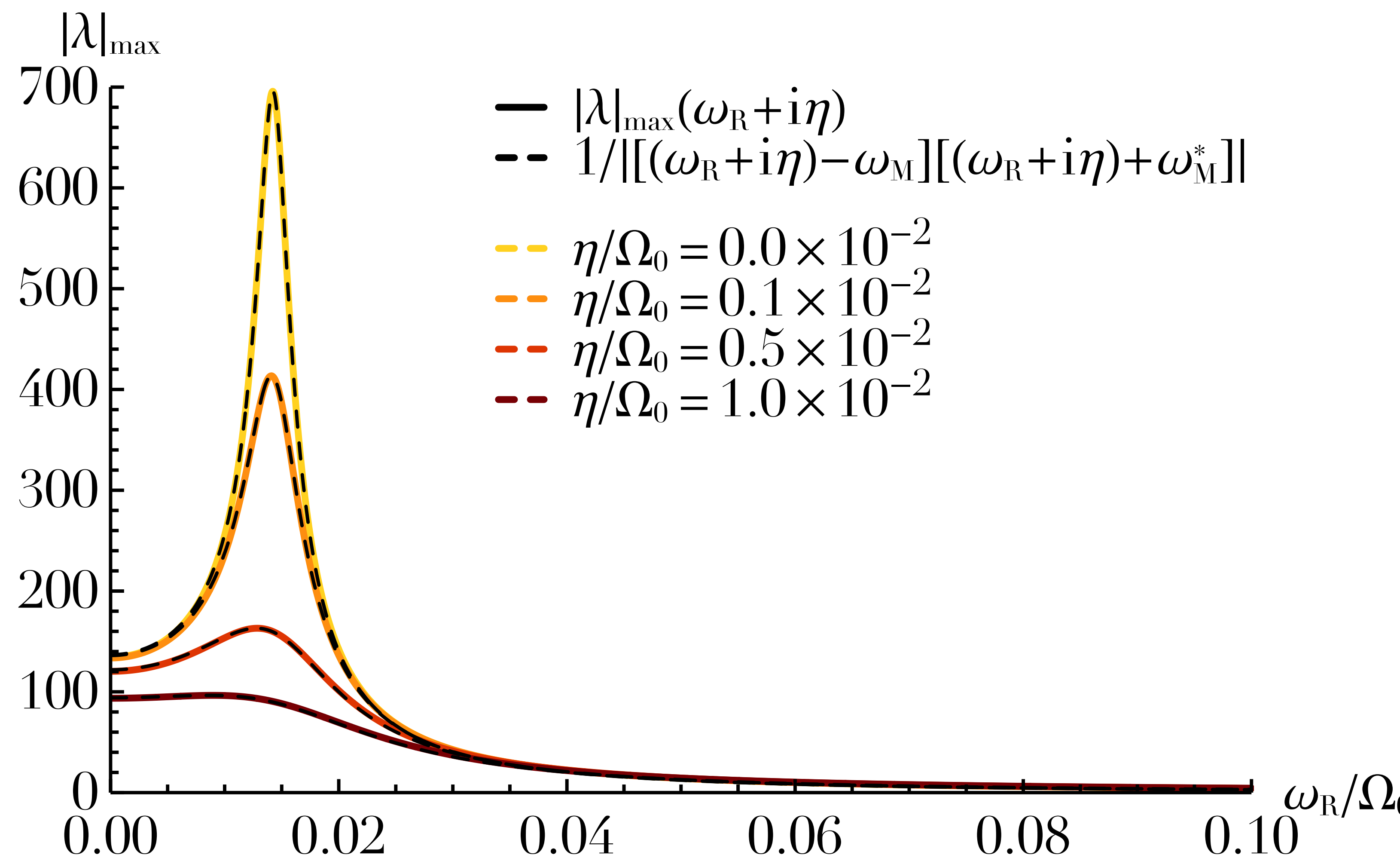
$$\big[\delta H(t)\big]_{\mathrm{trans.}} \simeq \mathrm{e}^{\gamma_{\mathrm{M}} t}$$

# How strong is the amplification?

Amplification eigenvalue

$$\lambda(\omega) = \mathrm{EigMax}\left[1/\boldsymbol{\varepsilon}(\omega)\right]$$




``Natural'' and symmetric ansatz

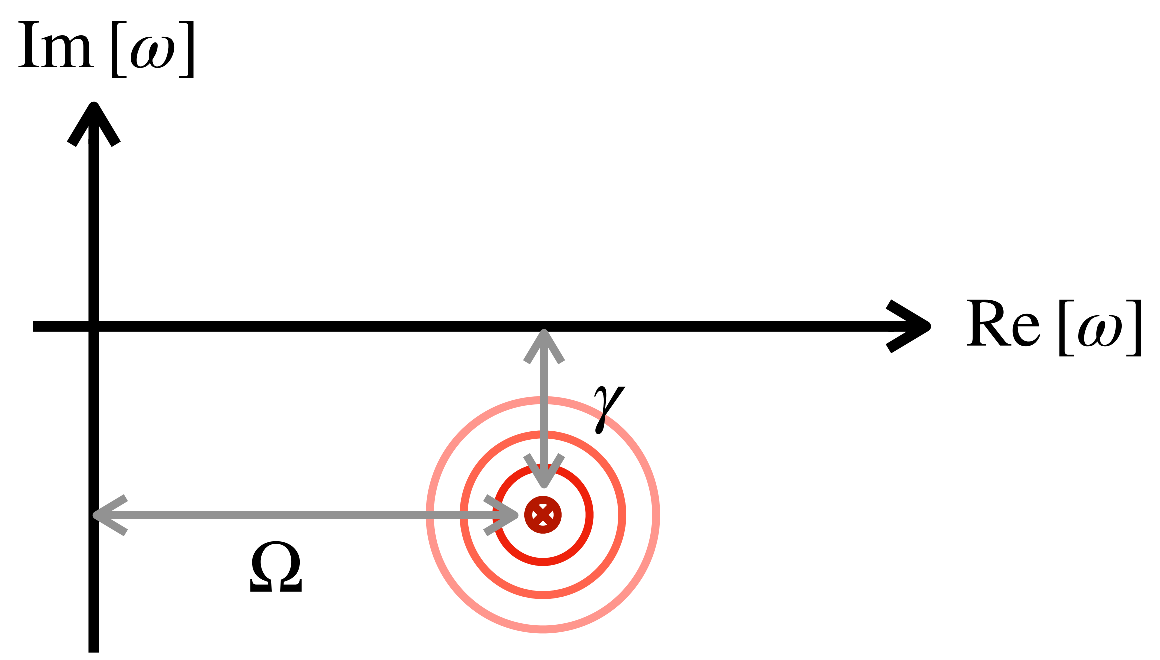
$$\lambda(\omega) \propto \frac{1}{|\,(\omega - \omega_{\mathrm{M}})(\omega + \omega_{\mathrm{M}}^{*})\,|}$$

**Why is such a simple ansatz so effective?**

# Weakly damped modes and Landau's trick

Root of the **dispersion function**

$$\varepsilon\big(\Omega + \mathrm{i}\,\gamma\big) = 0$$




The mode is **weakly damped** $\gamma \ll \Omega$

$$\varepsilon(\Omega) + \mathrm{i}\,\gamma \frac{\partial}{\partial\Omega}\varepsilon(\Omega) = 0$$

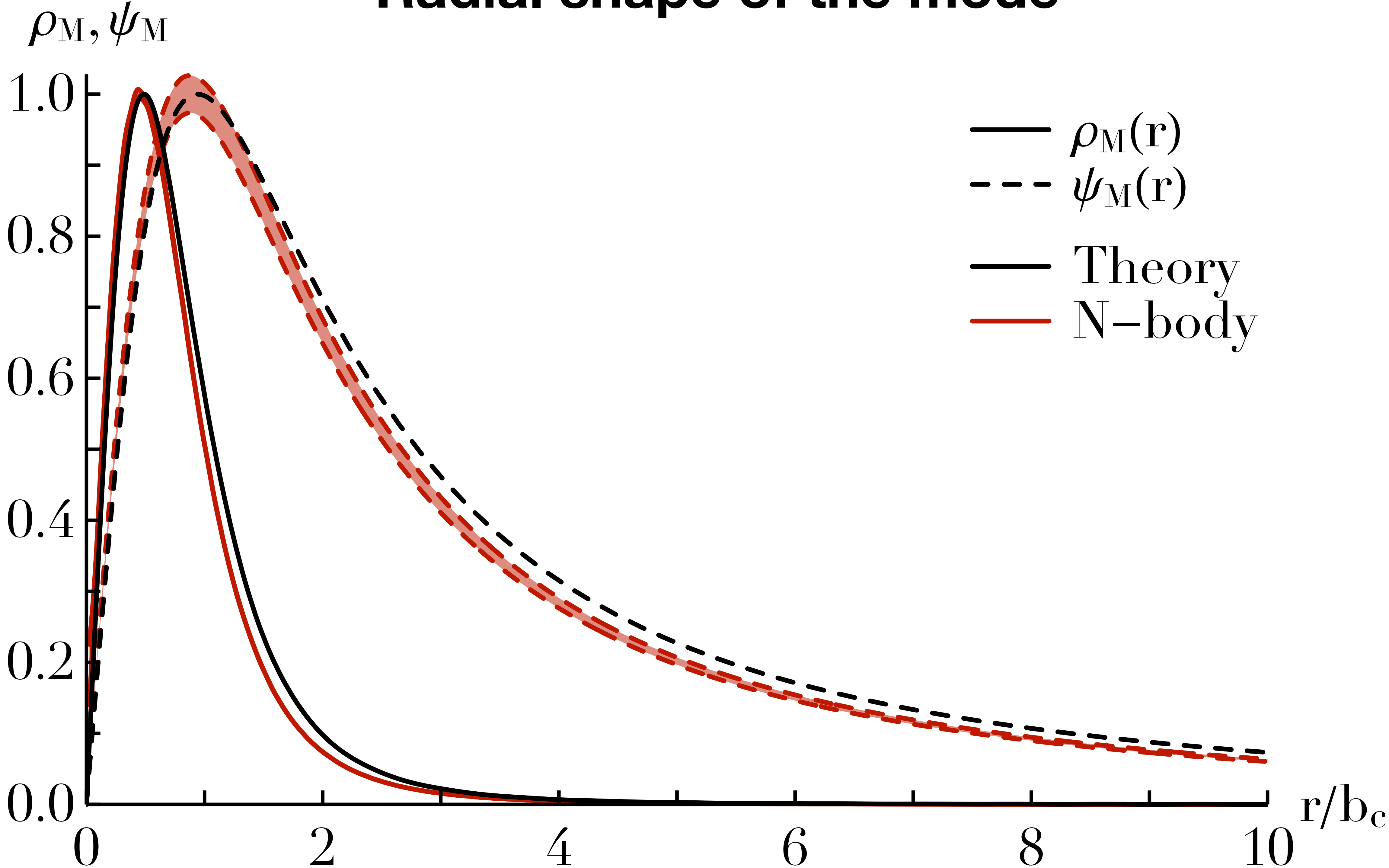
**Self-consistent constraints** for the mode's frequency

$$\mathrm{Re}\big[\varepsilon(\Omega)\big] = 0$$

$$\gamma = -\frac{\mathrm{Im}\big[\varepsilon(\Omega)\big]}{\partial\varepsilon(\Omega)/\partial\Omega}$$

**Can one infer the modes without ever going in the lower half of the complex plane?**

# Radial shape of the mode




To estimate the mode's shape from **N-body simulations**

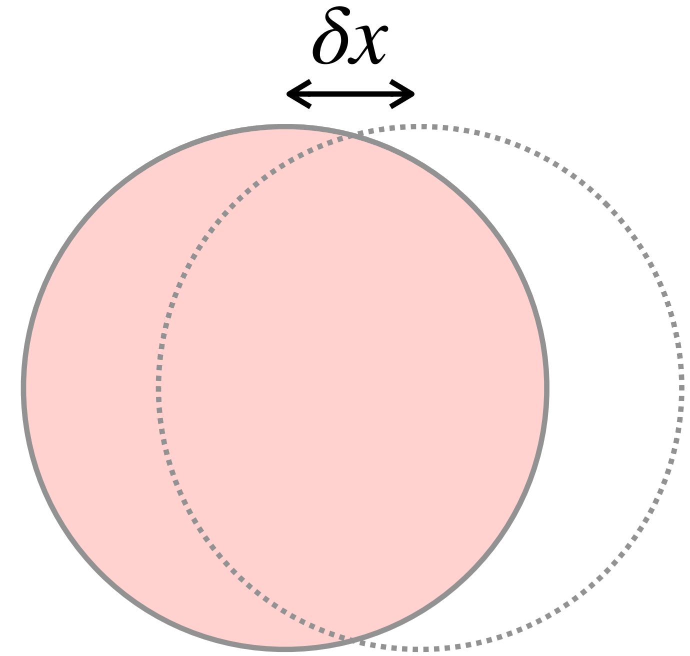
**Radial shell** projection for $\rho_M(r)$

*Heggie+(2020)*

**Multipole** projection for $\psi_M(r)$

*Lau+(2020)*

# Mode vs overall shift




**Whole cluster shift** by $\delta x$

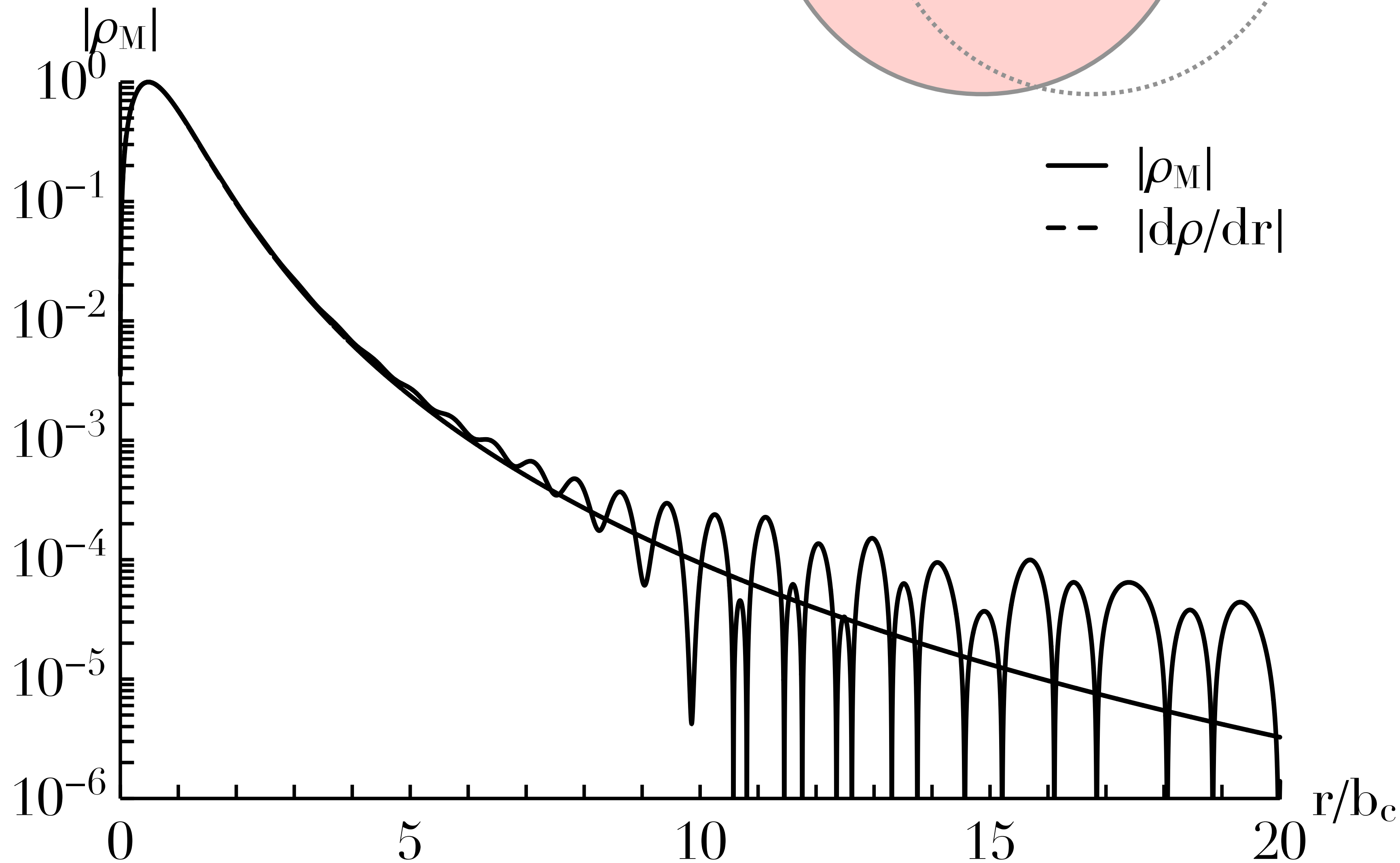
$$\delta\rho(x,y,z) = \rho_0(x-\delta x,y,z) - \rho_0(x,y,z)$$
$$= \frac{\mathrm{d}\rho_0}{\mathrm{d}r}\frac{x}{r}\,\delta x$$

Shift's **perturbation**

$$\delta\rho \propto \frac{\mathrm{d}\rho_0}{\mathrm{d}r}\,Y_{\ell m}(\hat{\mathbf{r}})$$

Mode's **perturbation**

$$\delta\rho \propto \rho_{\mathrm{M}}(r)\,Y_{\ell m}(\hat{\mathbf{r}})$$




**Why is the mode so similar to the density gradient?**

# Constraints on the radial shape

Conserving the **linear momentum**

$$\delta \mathbf{P} = \int \mathrm{d}\mathbf{r}\, \mathbf{r}\, \delta\rho(\mathbf{r}, t) = 0 \quad \Longrightarrow \quad \int \mathrm{d}r\, r^3 \rho_{\mathrm{M}}(r) = 0$$

Mode's **node**

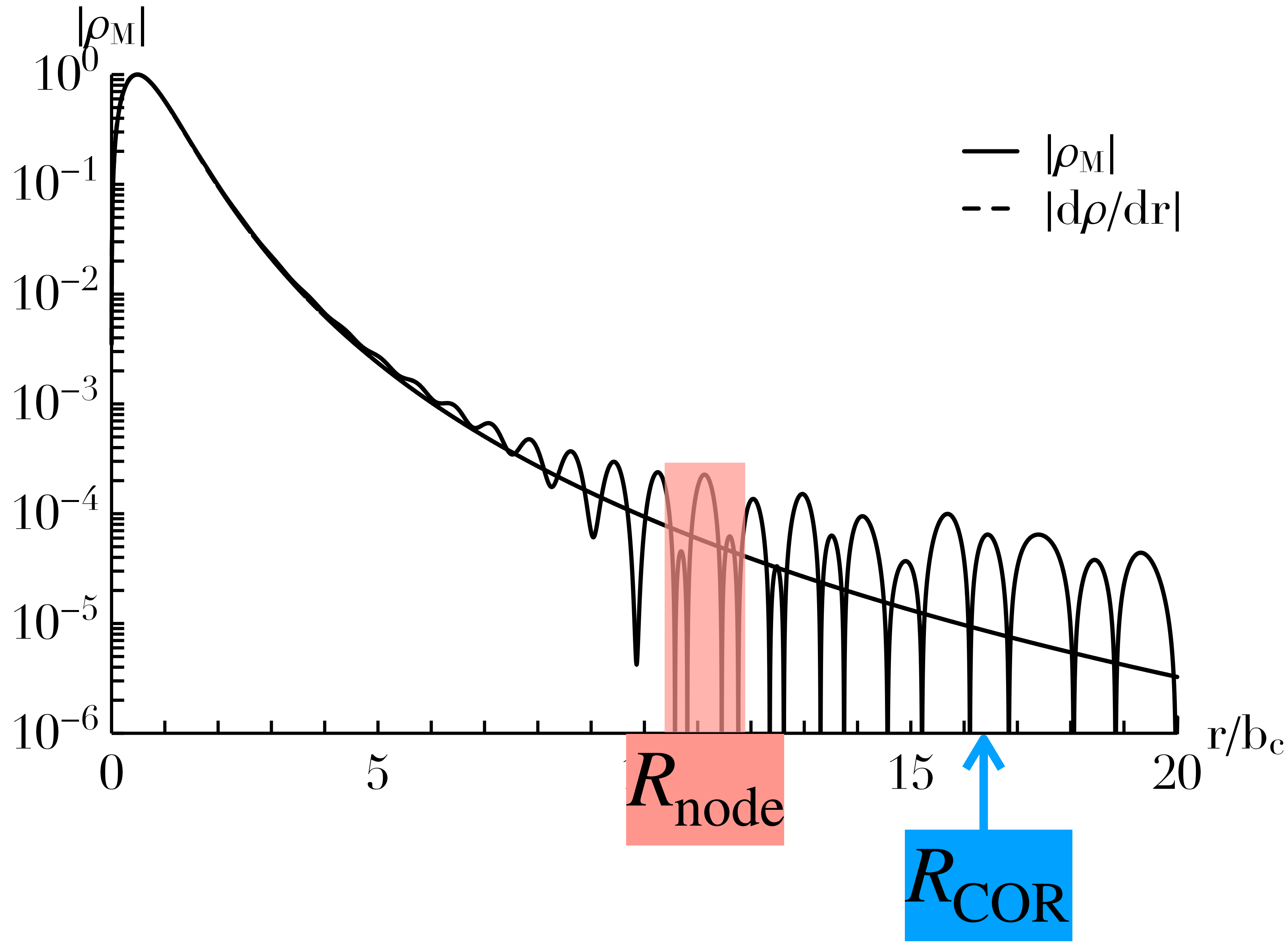
$$\rho_{\mathrm{M}}(R_{\mathrm{node}}) = 0$$

Mode's **corotation radius**

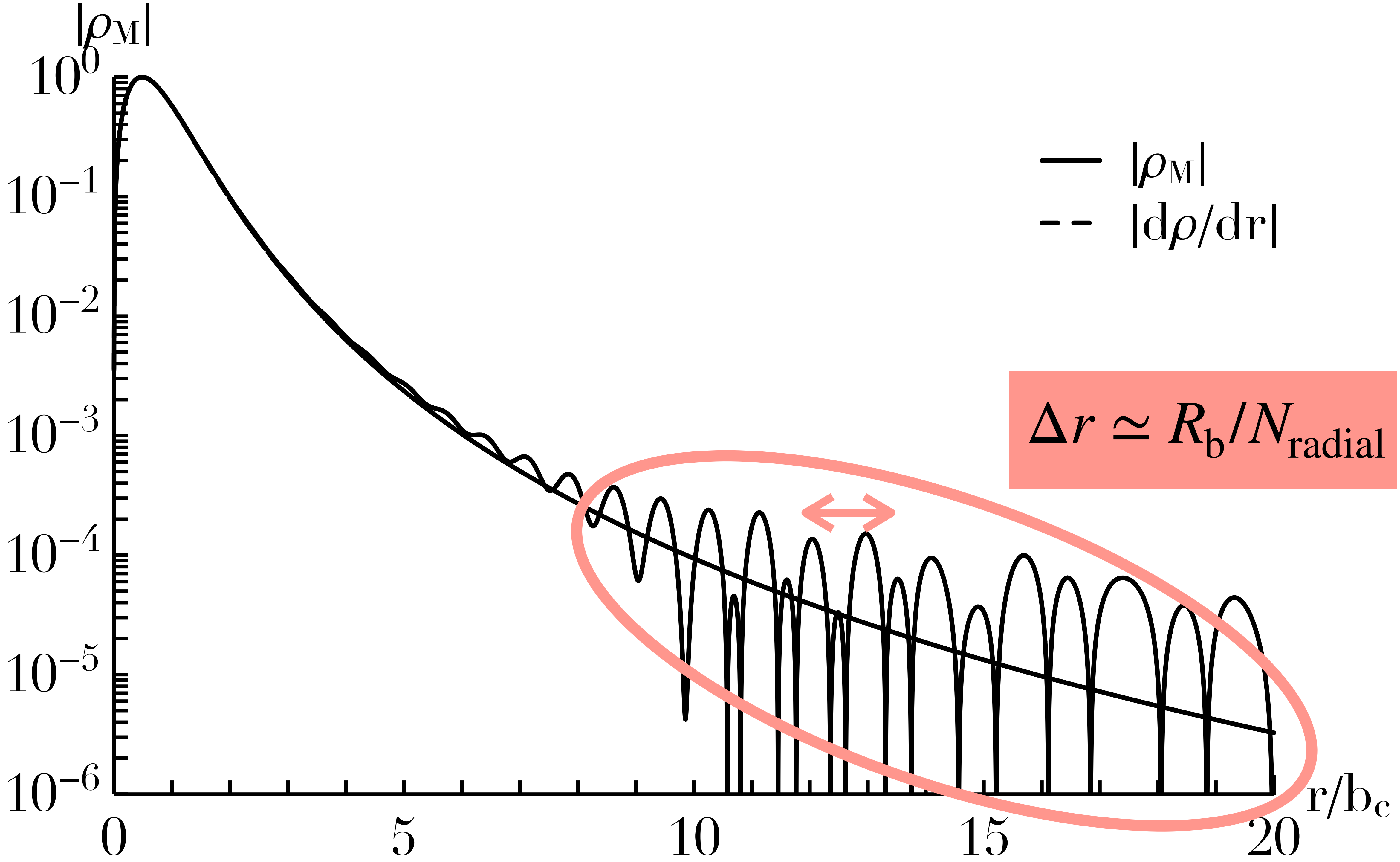
$$\Omega_2^{\mathrm{circ}}(R_{\mathrm{COR}}) = \mathrm{Re}[\omega_{\mathrm{M}}]$$

Constraint on **rotation**

$$R_{\mathrm{node}} \leq R_{\mathrm{COR}}$$



# Mode's node



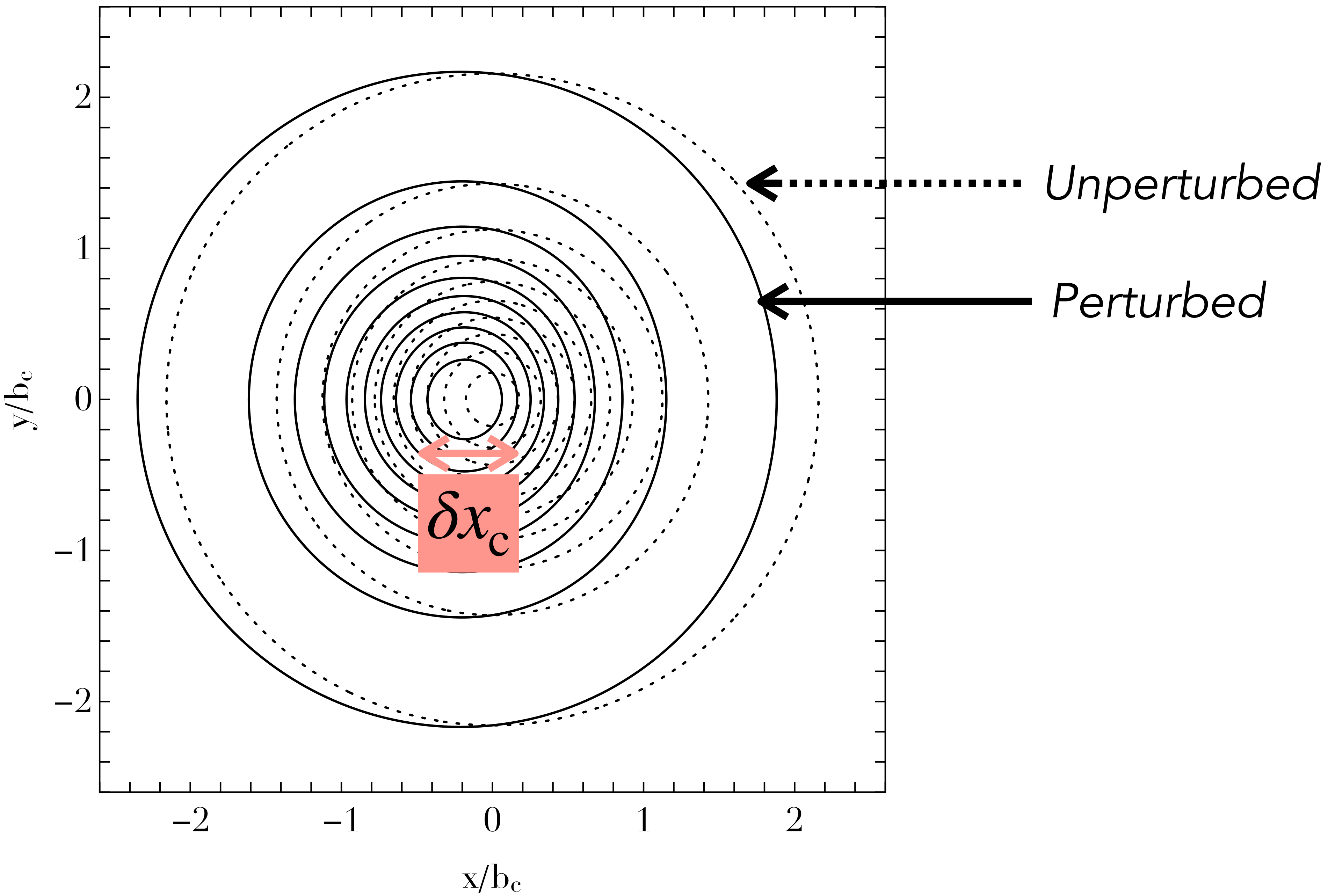

**How to reduce spurious oscillations from the basis elements?**

# Dynamics of the perturbation

Typical perturbation

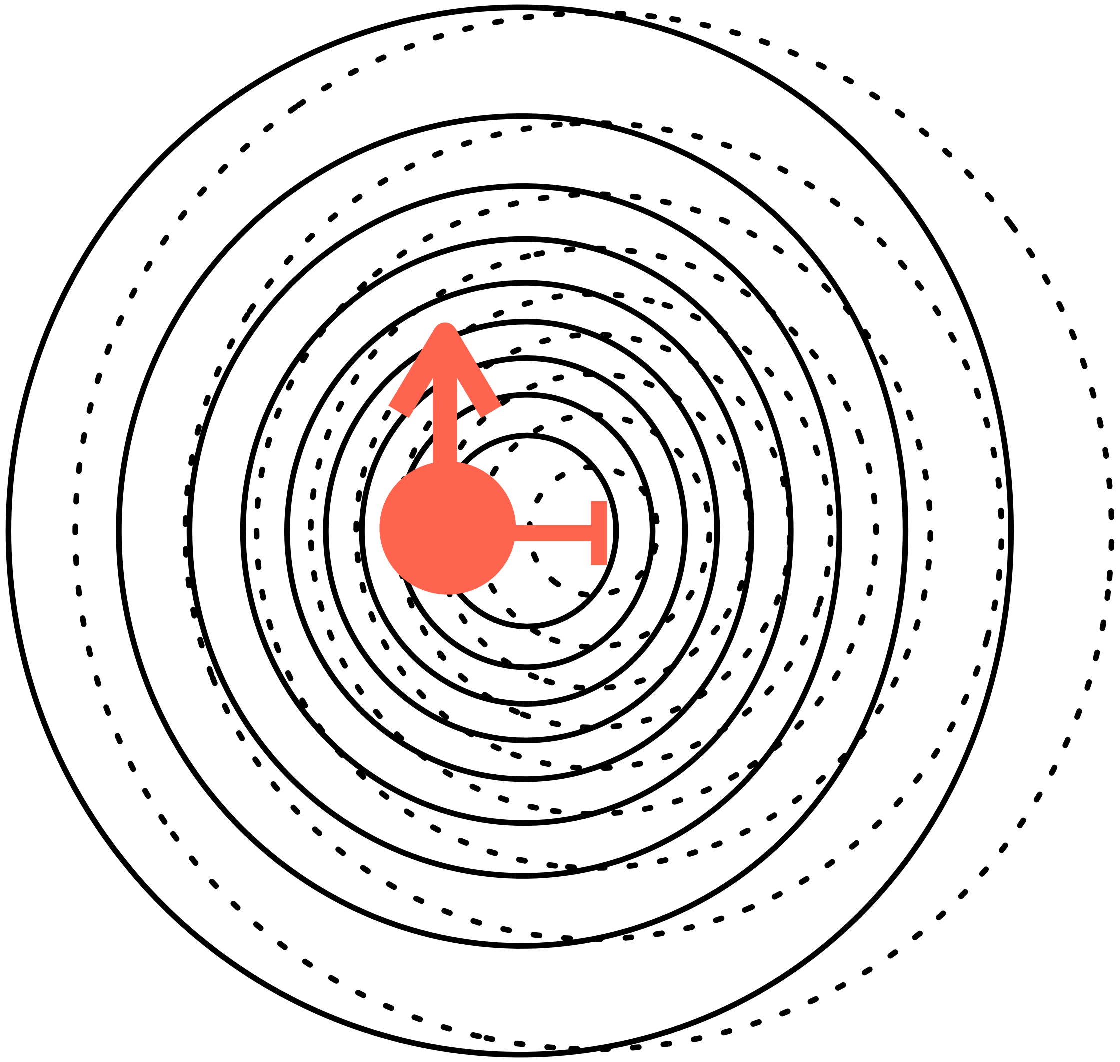
$$\delta\rho(\mathbf{r},t) = A_{\mathrm{M}}(t)\,\rho_{\mathrm{M}}(r)\,Y_{\ell m}(\widehat{\mathbf{r}})$$

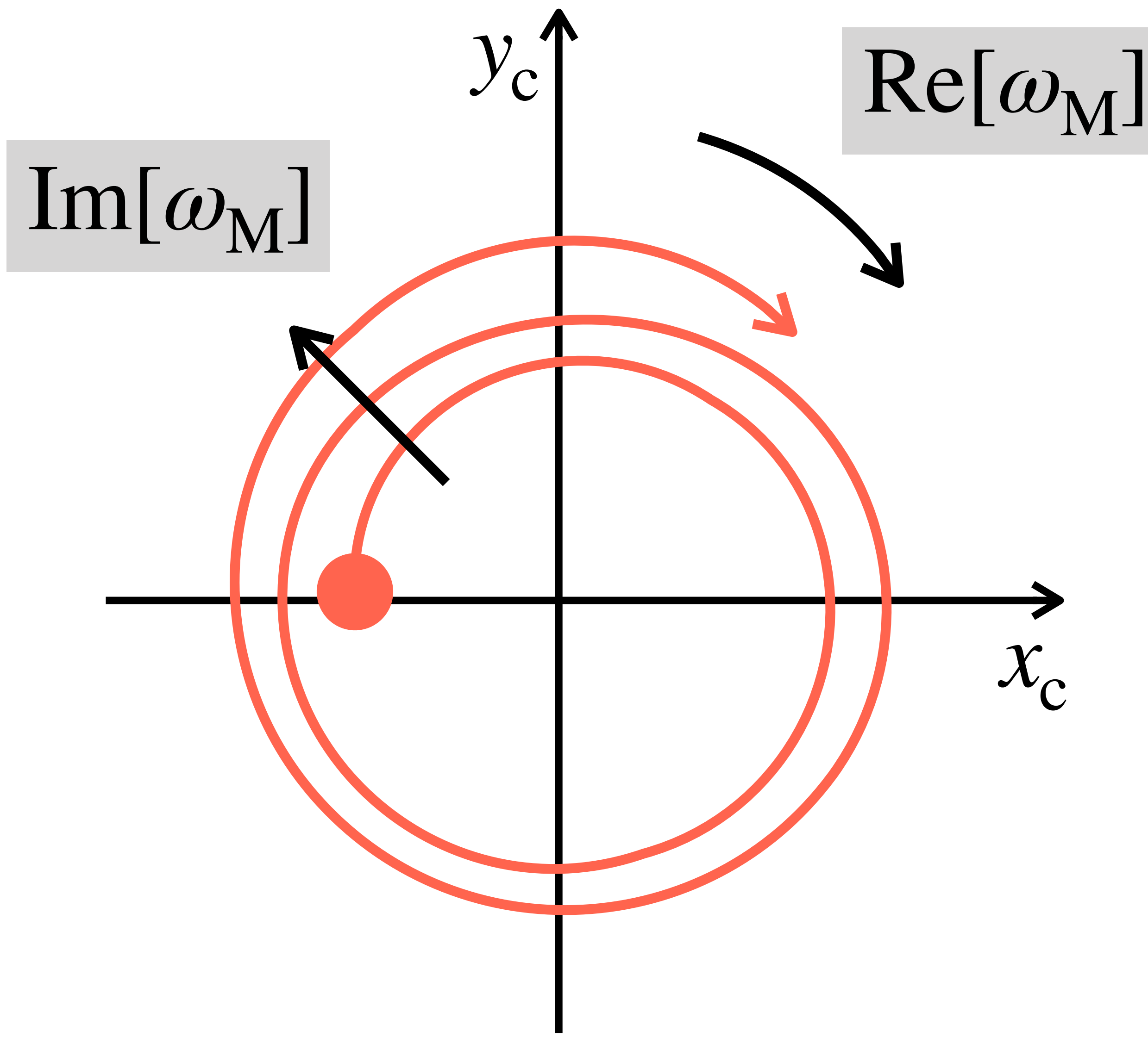
Wobble of the **density centre**



# Dynamics of the density centre

**Time evolution** of the density centre






**Rotation** timescale

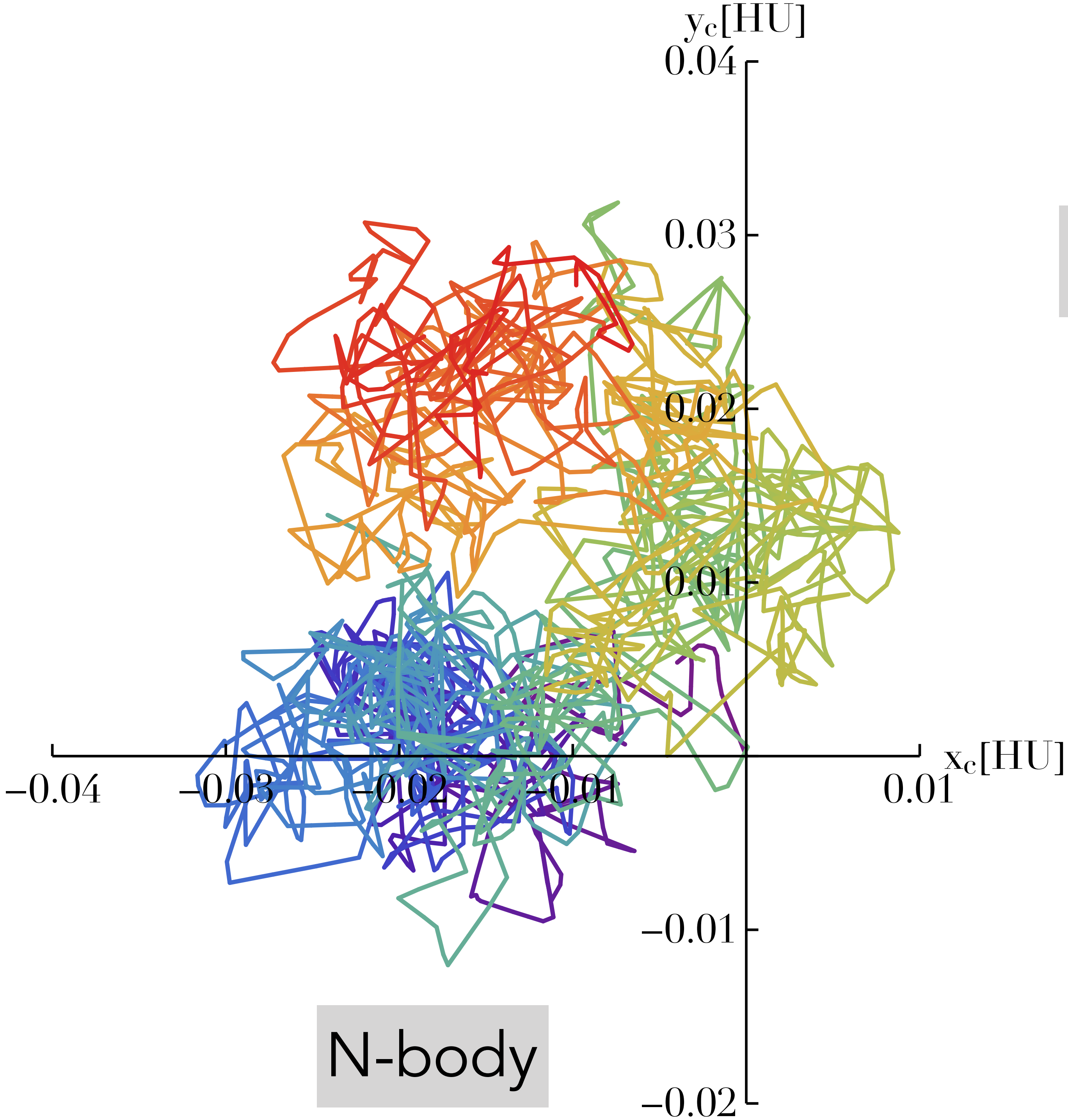
$$T = 2\pi/\mathrm{Re}[\omega_M] \simeq 50\,\mathrm{HU}$$

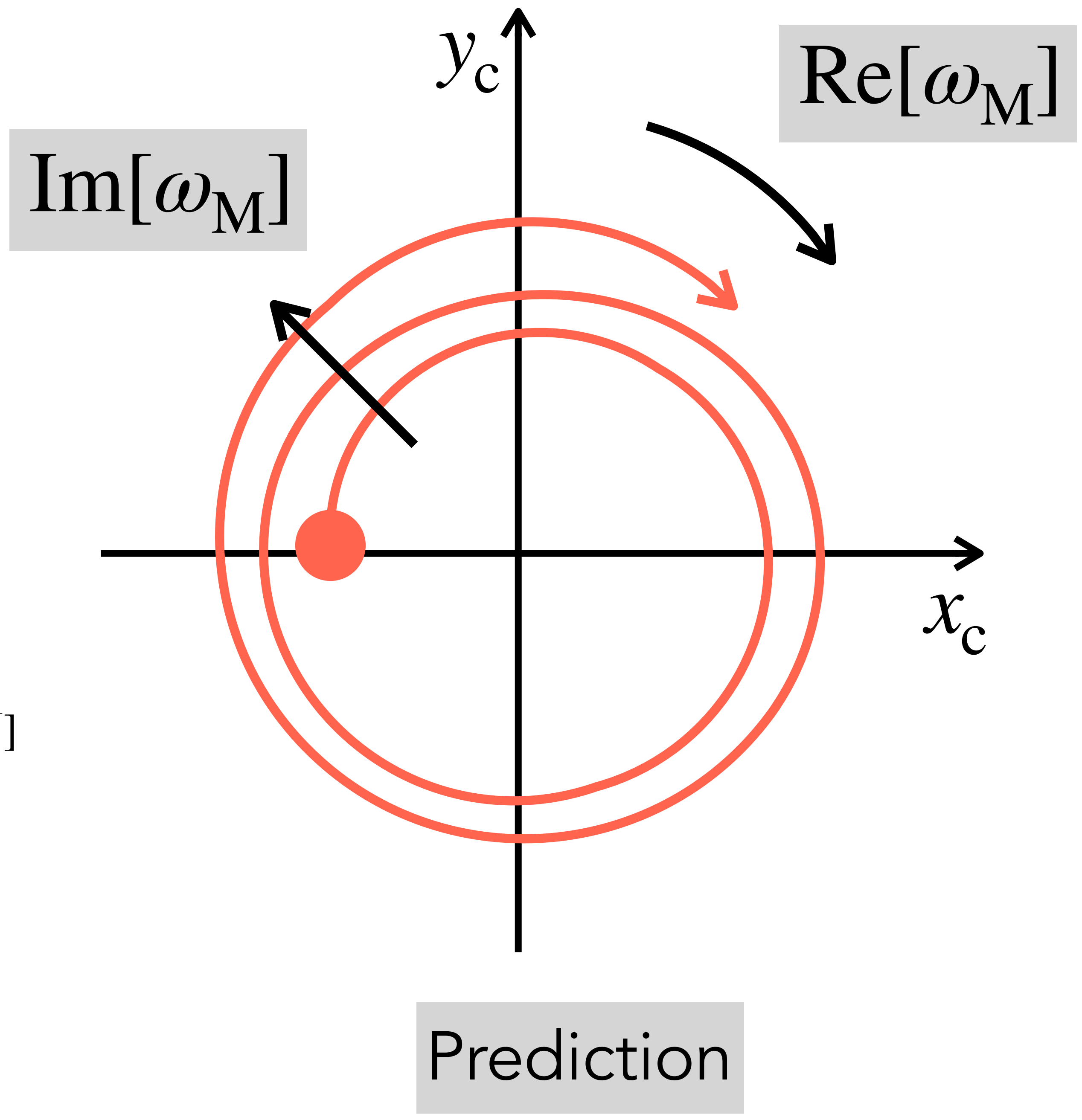
**Growth** timescale

$$T = 3/\mathrm{Im}[\omega_M] \simeq 250\,\mathrm{HU}$$

# Dynamics of the density centre

**Time evolution** of the density centre



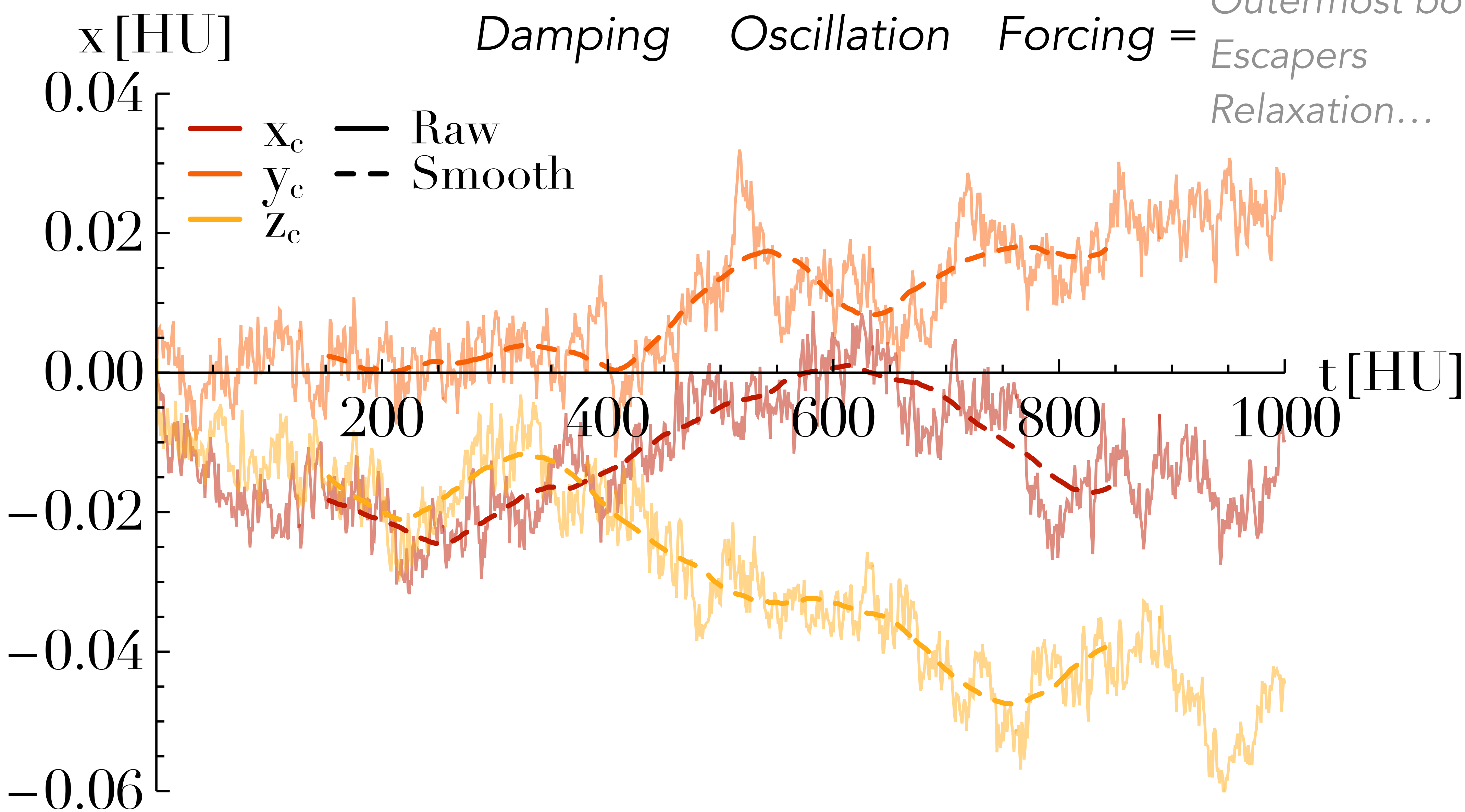

**Why is the centre's dynamics so messy?**

# Dynamics of the density centre

**Stochastic** dynamics

$$\frac{d^2 x_c}{dt^2} - \gamma_M \frac{d x_c}{dt} + \Omega_M^2 \, x_c = \eta(t)$$

*Damping* *Oscillation* *Forcing* = *Perpetual Poisson noise* / *Outermost bound particles* / *Escapers* / *Relaxation…*



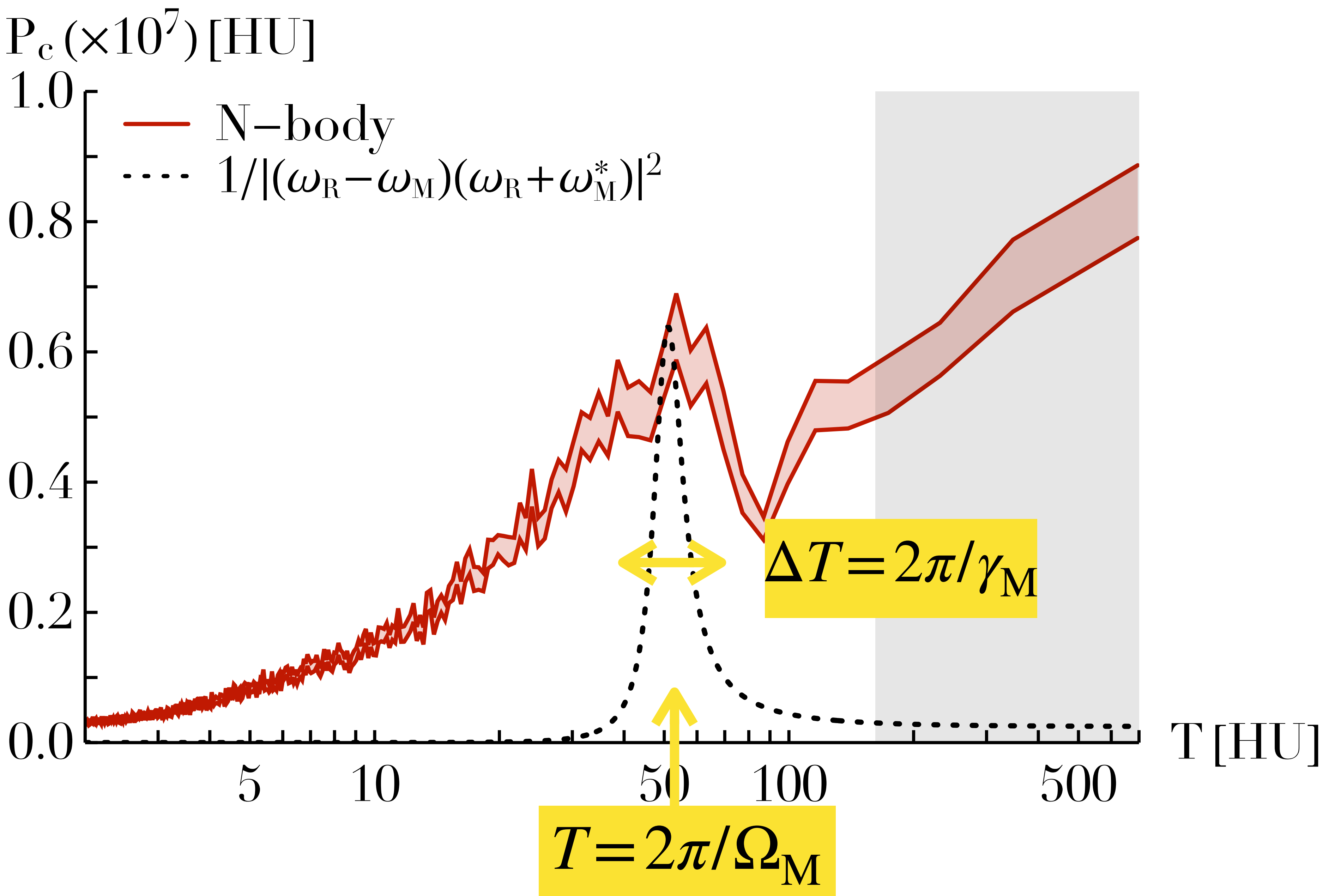

**How to understand these large-scale excursions?**

# Dynamics of the density centre

## Power spectrum

$$\left\langle \left| \hat{x}_{\mathrm{c}}(\omega) \right|^2 \right\rangle \propto \frac{1}{\left| (\omega - \omega_{\mathrm{M}})\,(\omega + \omega_{\mathrm{M}}^{*}) \right|^2}$$

In **N-body simulations** *(with time-filtering)*



# What about QL diffusion?

# Long-term dynamics

Decomposing the **fluctuations**

$$\delta\Phi_{\mathrm{tot}}(t) = \delta\Phi_{\mathrm{BL}}(t) + \delta\Phi_{\mathrm{M}}(t)$$

| Total fluctuations | Drives BL | Drives QL |
|---|---|---|

Two sources of **evolution**

$$\frac{\partial F}{\partial t} = \left(\frac{\partial F}{\partial t}\right)_{\mathrm{BL}} + \left(\frac{\partial F}{\partial t}\right)_{\mathrm{QL}}$$

Requires a **splitting** of the perturbations

$$\left\{\mathbf{x}_i(t), \mathbf{v}_i(t)\right\} \mapsto \left\{\delta\Phi_{\mathrm{BL}}(t), \delta\Phi_{\mathrm{M}}(t)\right\}$$

**How to measure the waves' amplitude in N-body runs?**

# Mode's energy

Typical perturbation

$$\delta\rho(\mathbf{r},t) = A_{\mathrm{M}}(t)\,\rho_{\mathrm{M}}(r)\,Y_{\ell m}(\hat{\mathbf{r}})$$

**Energy** in the mode

$$E_{\mathrm{M}}(t) = |A_{\mathrm{M}}(t)|^2$$

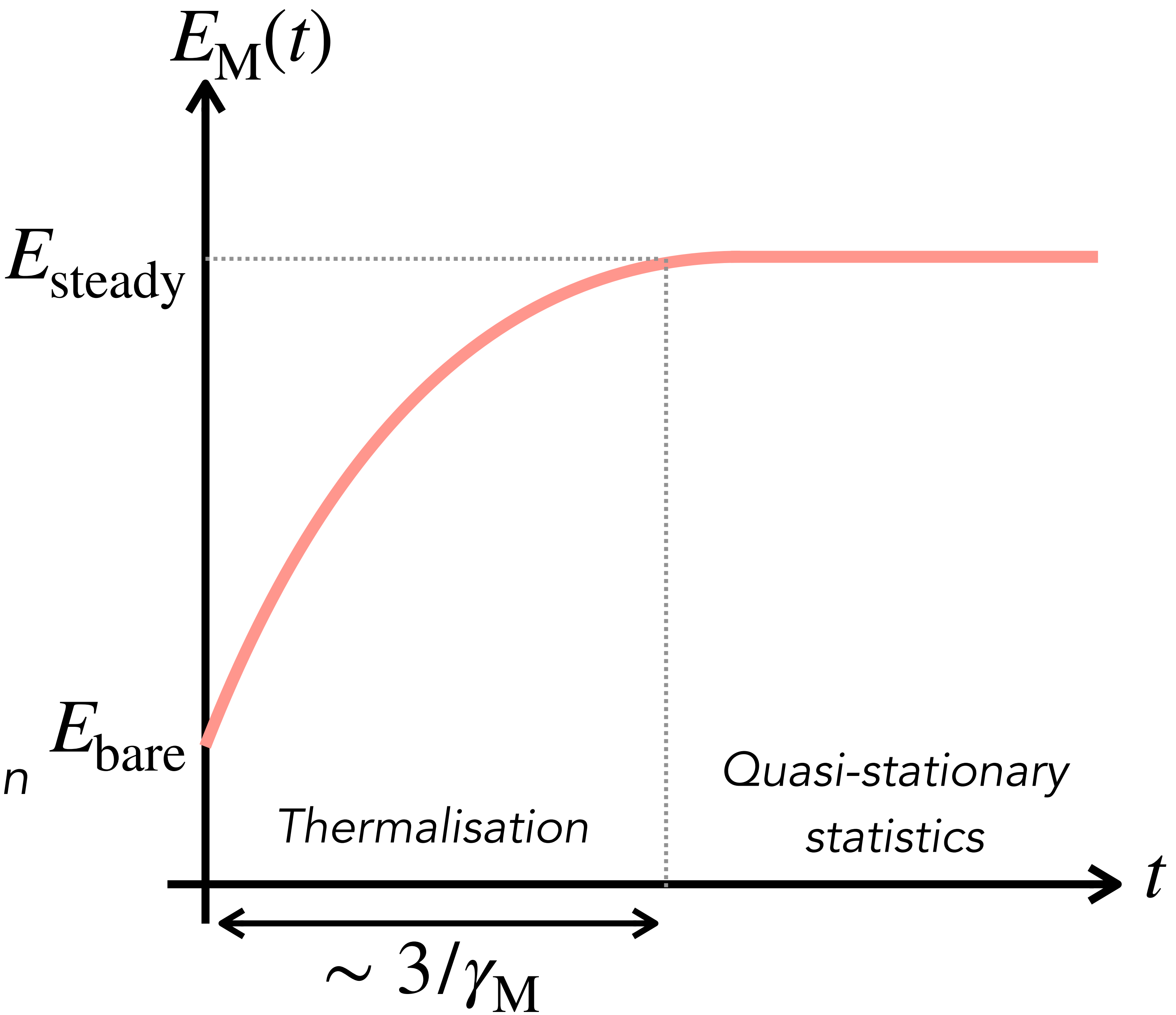
**Wave equation** *Hamilton+(2020)*

$$\frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} = 2\gamma_M\,E_{\mathrm{M}} + S_{\mathrm{M}}$$

Energy budget

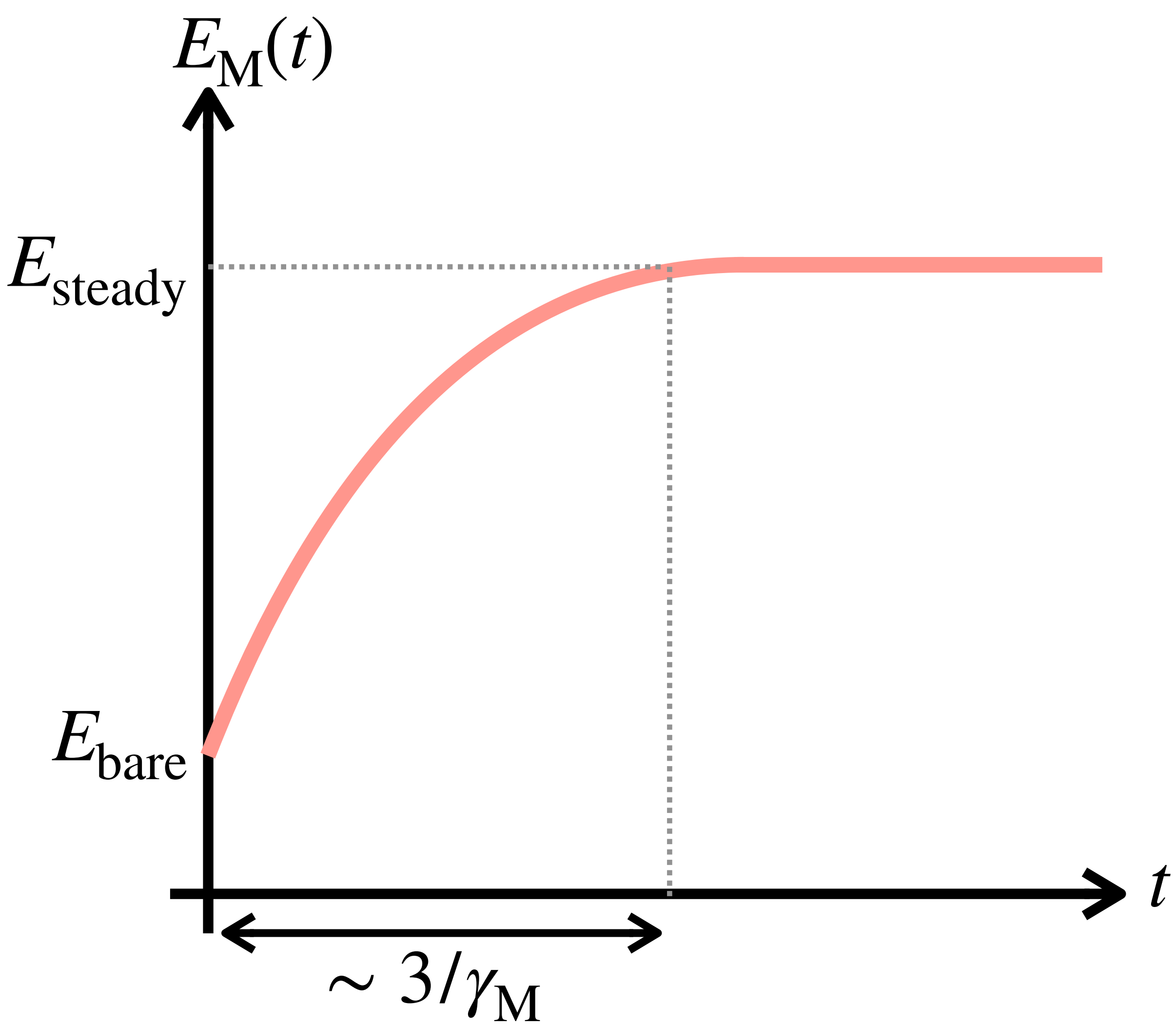
$2\gamma_{\mathrm{M}}\,E_{\mathrm{M}}$ **Landau damping** *Resonant interaction*

$S_{\mathrm{M}}$ **Spontaneous emission** *Perpetual Poisson noise*

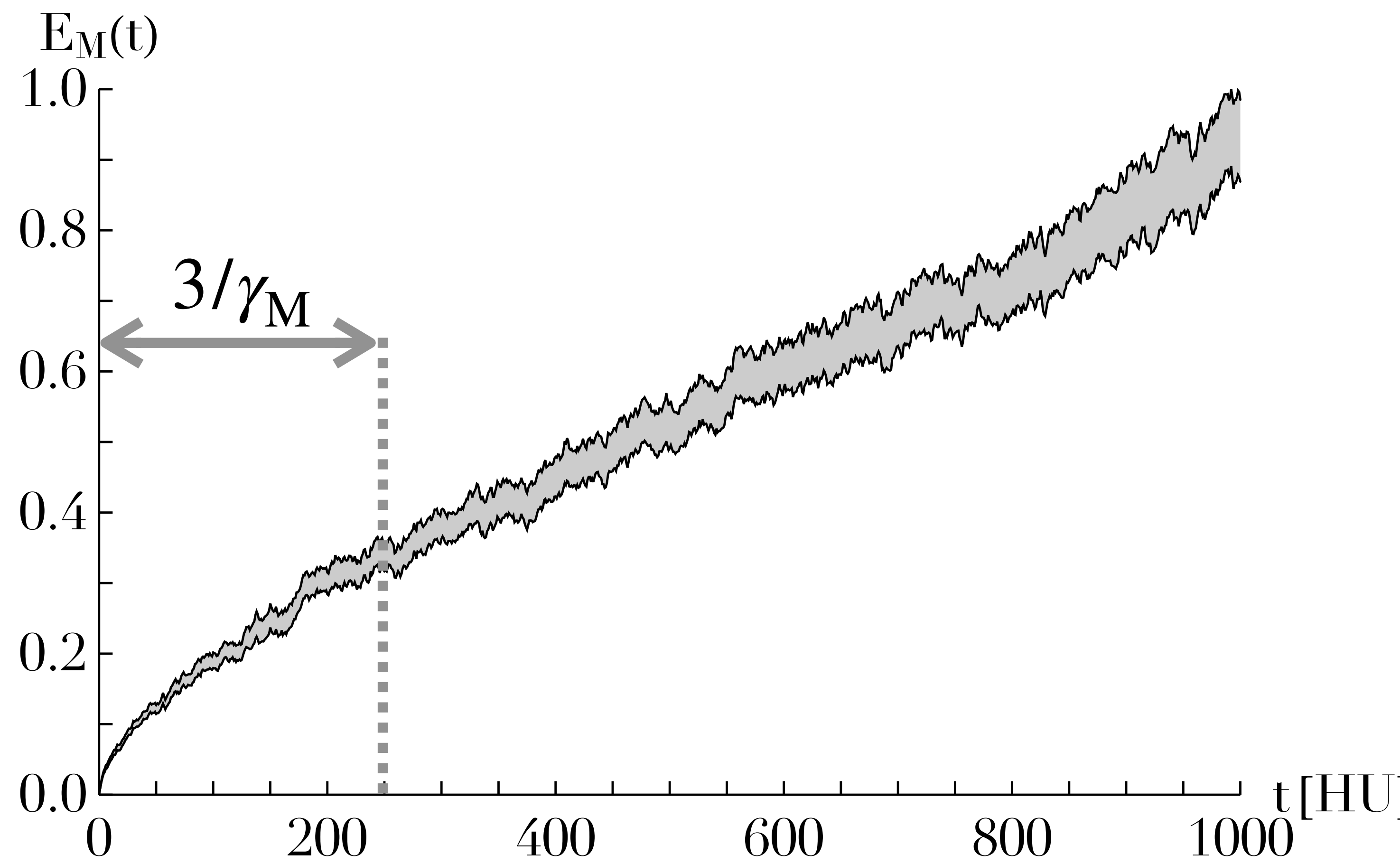


# Mode's energy

Prediction




N-body simulations




**No saturation**

Diffusive-like dynamics

**How to check the wave equation in N-body simulations?**

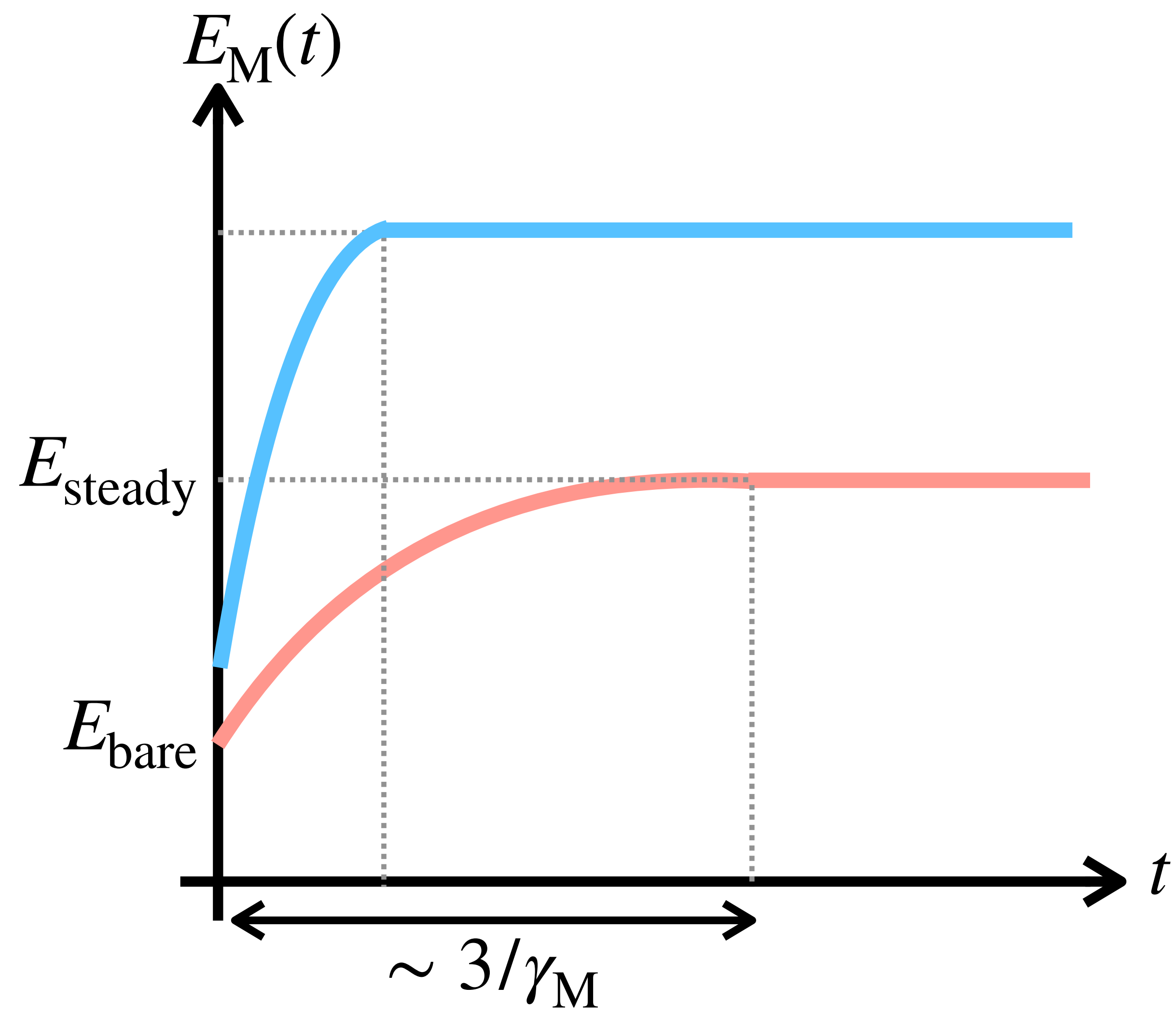
# Dominating mode

## Wave equation

$$\frac{\mathrm{d}E_{\mathrm{M}}}{\mathrm{d}t} = 2\gamma_M \, E_{\mathrm{M}} + S_{\mathrm{M}}$$

## Steady energy

$$E_{\mathrm{steady}} = -\frac{S_{\mathrm{M}}}{2\gamma_{\mathrm{M}}}$$




## Spontaneous emission

$$S_{\mathrm{M}} = \frac{1}{N}\sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J}\, \delta_{\mathrm{D}}[\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \Omega_{\mathrm{M}}]\, F(\mathbf{J})$$

**Which mode is dominating?**

# What happens after thermalisation?

(Weakly damped) **Quasilinear Theory** *Hamilton+(2020)*

$$\frac{\partial F(\mathbf{J})}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \left[ \sum_{\mathbf{k}} \mathbf{k}\, \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \Omega_{\mathrm{M}}) \left\{ \frac{1}{N} F(\mathbf{J}) - E_{\mathrm{M}}(t)\, \mathbf{k}\cdot\frac{\partial F}{\partial \mathbf{J}} \right\} \right]$$

$$\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \Omega_{\mathrm{M}})$$

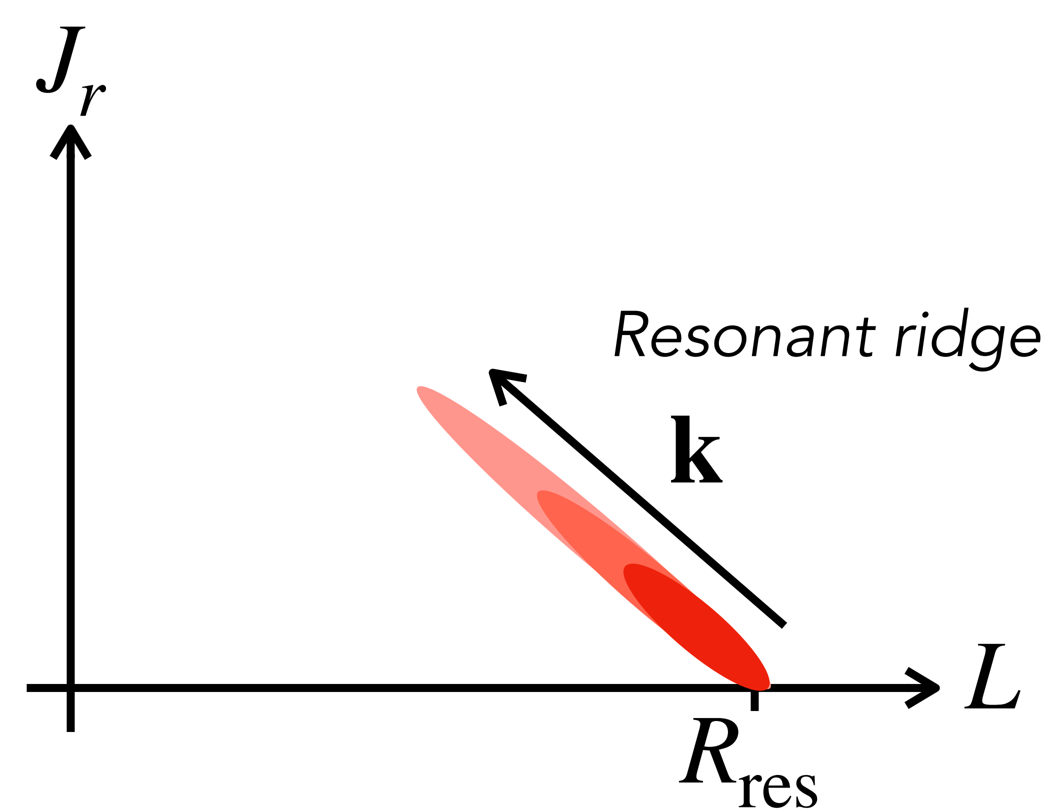
*Resonant* ***particle-wave*** *interaction*

$F(\mathbf{J})$ — ***Friction***

*Cost of spontaneous emission*

$E_{\mathrm{M}}(t)\, \mathbf{k}\cdot\dfrac{\partial F}{\partial \mathbf{J}}$ — ***Diffusion***

*Resonant absorption*




**How to integrate over the QL resonance condition?**

# BL vs QL

## Balescu-Lenard equation

$$\frac{\partial F(\mathbf{J})}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[ \sum_{\mathbf{k},\mathbf{k}'} \int \mathrm{d}\mathbf{J}' \, \frac{\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\mathbf{\Omega}(\mathbf{J})')}{|\,\varepsilon(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}))\,|^2} \times \cdots \right]$$

## QL diffusion equation

$$\frac{\partial F(\mathbf{J})}{\partial t} = \frac{1}{N}\frac{\partial}{\partial \mathbf{J}} \cdot \left[ \sum_{\mathbf{k}} \mathbf{k}\, \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \Omega_{\mathrm{M}}) \times \cdots \right]$$

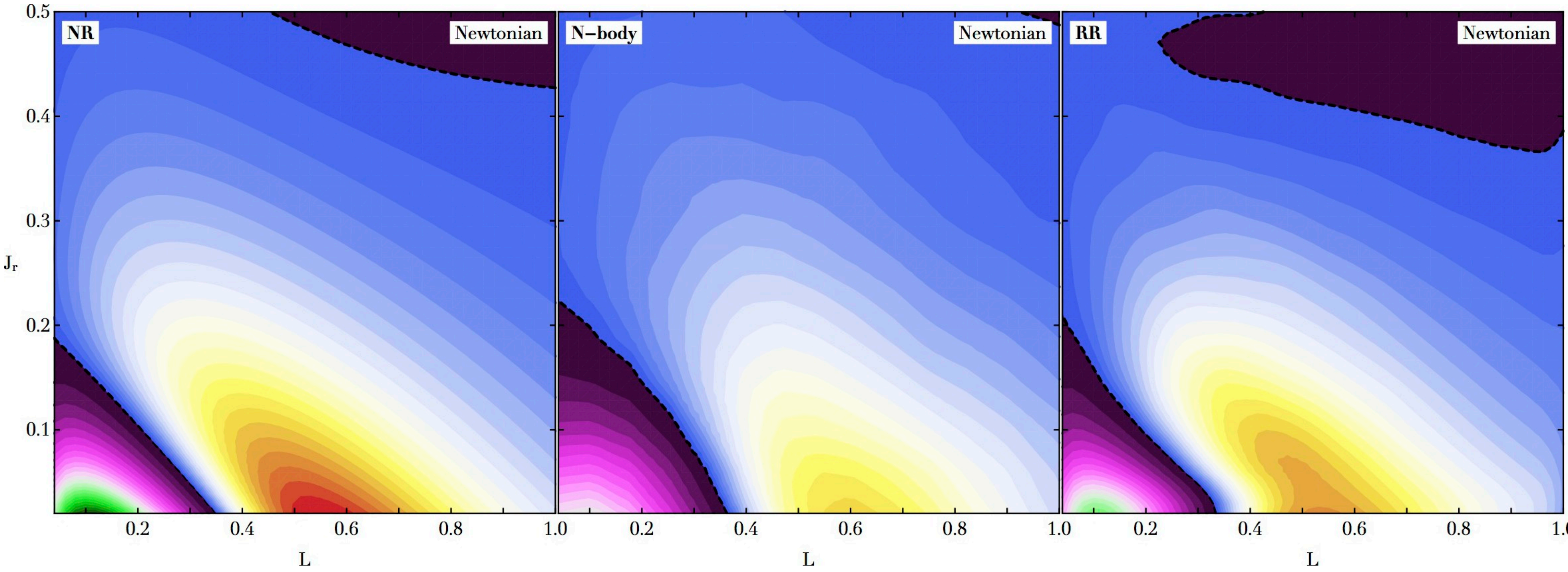
Close to a (weakly) **damped mode**

$$\mathrm{Im}[\omega_{\mathrm{M}}] \ll \mathrm{Re}[\omega_{\mathrm{M}}] \quad \Longrightarrow \quad \frac{1}{|\,\varepsilon(\Omega_{\mathrm{M}})\,|} \gg 1$$

**Can QL ever dominate over BL?**

# Long-term relaxation

Diffusion in **orbital space** $\partial F(\mathbf{J})/\partial t$




*Orbit-averaged homogeneous Landau*

*Direct N-body*

*Inhomogeneous Balescu-Lenard*

**Why don't we find any trace of QL in numerical simulations?**

# Conclusion

Alternative approach to **analytic continuation**

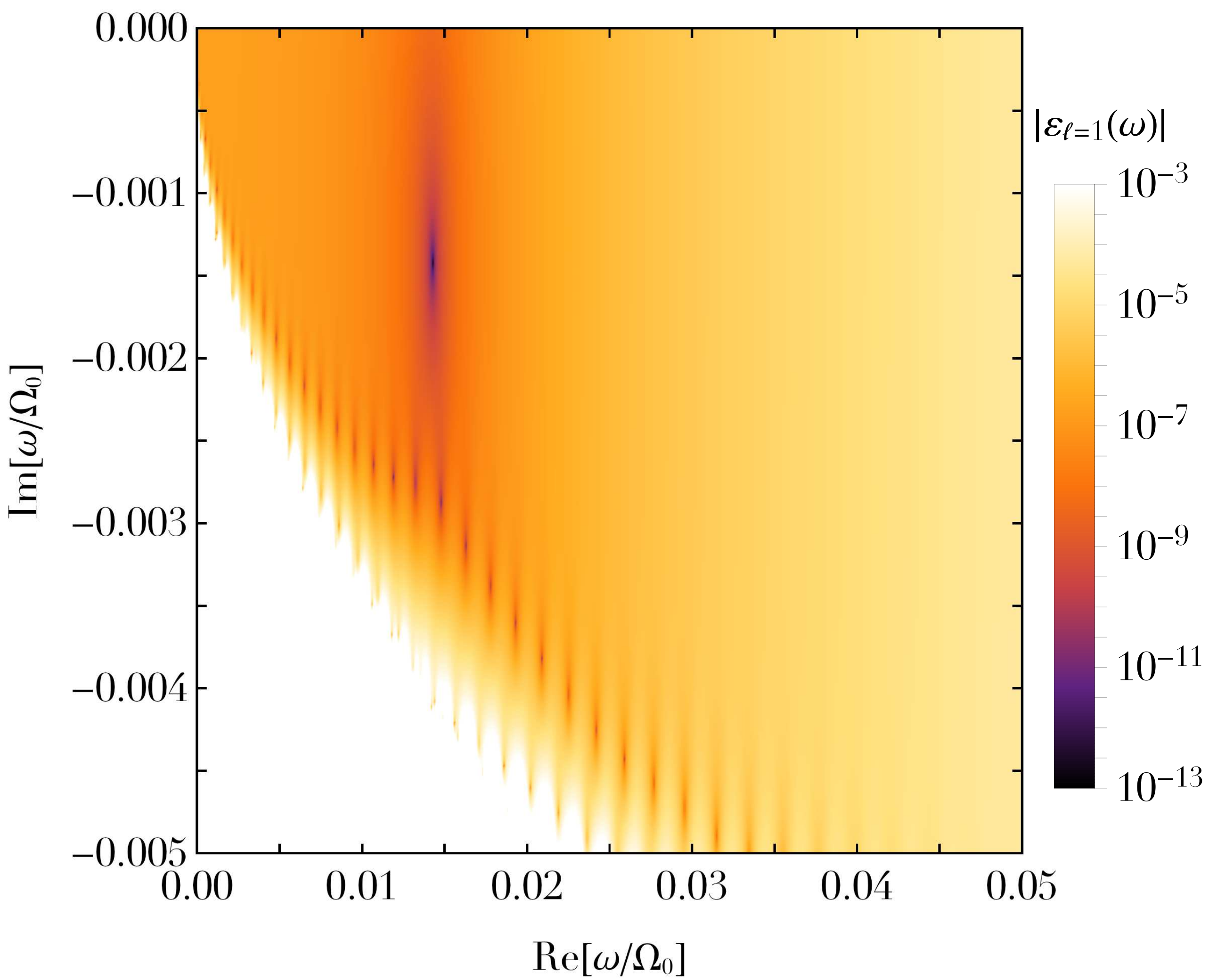
$$M(\omega) = \sum_k a_k D_k(\omega)$$

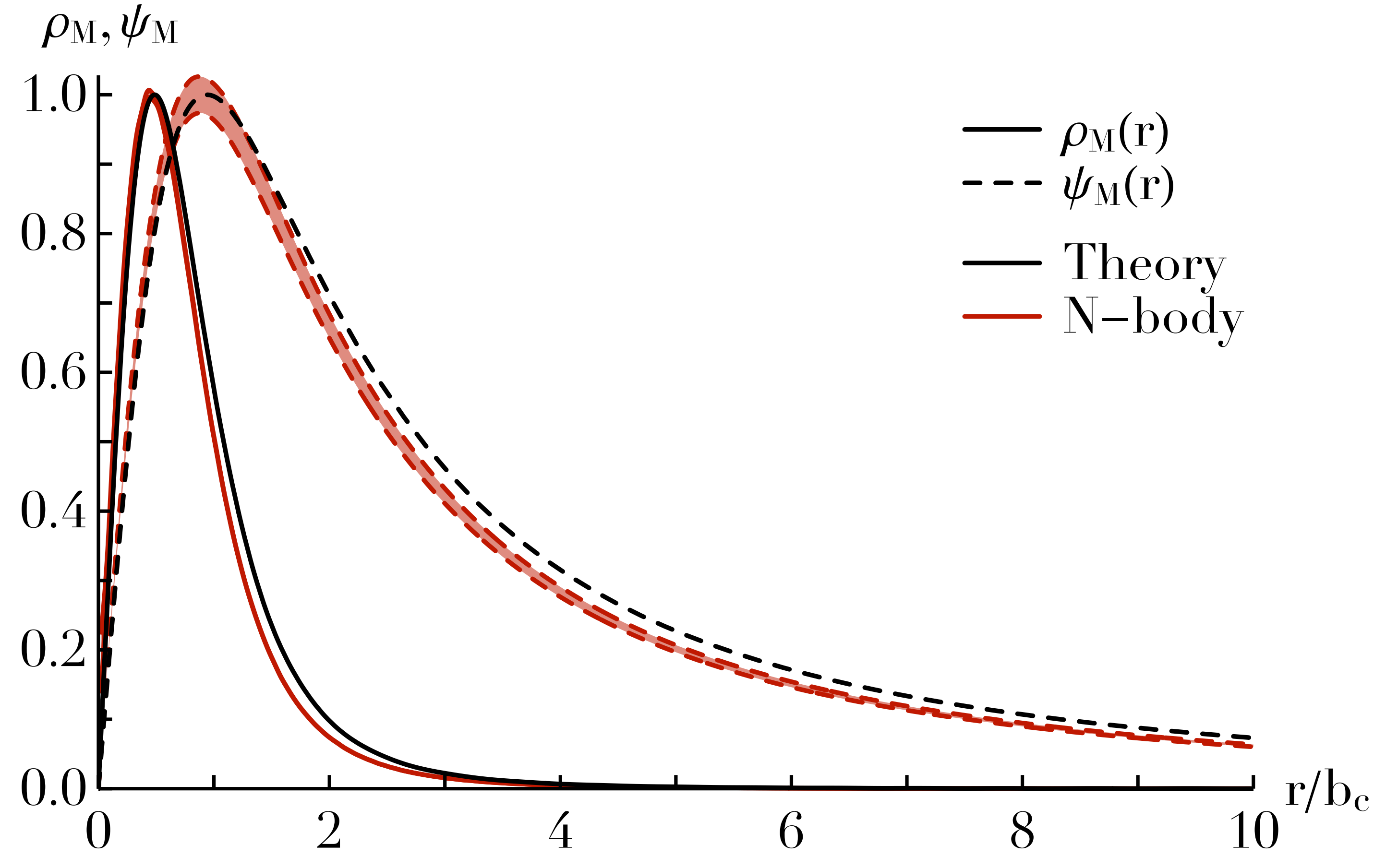
*Legendre series*

$$M(\omega) = \frac{P(\omega)}{Q(\omega)}$$

*Rational functions* Weinberg(1994)

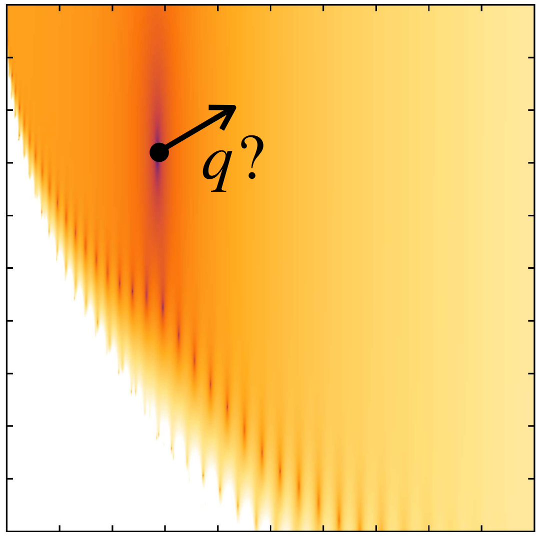
**(Weakly) damped modes** are unavoidable in globular clusters



# Future works

## Impact of **anisotropy**




## Impact of **potential**

Less puffy *(e.g., Plummer)*

Truncated *(e.g., King)*

Cuspy *(e.g., Hernquist)*

Degenerate *(e.g., quasi-Keplerian)*

## **Thermalisation** timescale

$1/\gamma_{\mathrm{M}}$ vs $F_{\mathrm{vK}}(\mathbf{J}, \omega)$ Lau+(2020)

*Landau* *van Kampen*

## **Others**

Other harmonics
*What is so special with* $\ell = 1$?

Disc dynamics
*Swing amplification*

QL theory and escapers