Self-gravitating systems and Balescu-Lenard equation

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Long-term relaxation

How do systems diffuse?

- Local Brownian diffusion
- Homogeneous Plasma diffusion
- Inhomogeneous Galaxy diffusion

Fluctuation-Dissipation Theorem

Diffusion ↔ Noise

Same process occur in galaxies, but:

Gravity is long-range
+ Stars follow orbits and resonate
+ Galaxies amplify perturbations

How do galaxies evolve on cosmic timescales?
The gravitational Balescu-Lenard equation

What does it require?

What is it?

Where does it come from?

Does it work?

What’s next?
What does the Balescu-Lenard Eq. require?
Galaxies are:

+ **Inhomogeneous** (complex trajectories)
+ **Relaxed** (equilibrium states)
+ **Resonant** (orbital frequencies)
+ **Degenerate** (in some regions)
+ **Self-gravitating** (amplification of perturbations)
+ **Discrete** (finite-N effects)
+ **Perturbed** (effects of the environment)

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What does it require?

Inhomogeneous

\[(x, v) \downarrow (\theta, J)\]

Angle-Action coordinates

Relaxed

\[F = F(J, t)\]

Quasi-stationary states

Resonant

\[\Omega(J) = \frac{\partial H_0}{\partial J}\]

Fast/Slow timescale

Self-gravitating

\[\frac{1}{|\varepsilon(\omega)|}\]

Linear response theory

Discrete & Perturbed

\[\frac{1}{N}\]

Finite-N effects
## Self-gravitating systems and Balescu-Lenard equation

### What does it require?

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- **Self-gravitating**
  - Linear response theory
  - \(\frac{1}{|\epsilon(\omega)|}\)

- **Discrete & Perturbed**
  - Finite-N effects
  - \(\frac{1}{N}\)
**Inhomogeneous systems**

+ **Label** orbits with **integrals of motion**

+ **Angle-Action coordinates**
  \[
  \begin{align*}
  \theta(t) &= \theta_0 + t \Omega(J) \\
  J(t) &= \text{cst.}
  \end{align*}
  \]

  Trajectories become straight lines

+ **Frequencies’ commensurability**: \( n \cdot \Omega(J) = 0 \)

+ **Relaxation**
  \[
  \text{few} t_{\text{cross}} \rightarrow F = F(J, t)
  \]
Example: Orbits in a disc

**Integrable orbits**

$$\Phi_0 = \Phi_0(R, z)$$

$$\left\{ \begin{array}{l}
\theta(t) = \theta_0 + t \Omega(J) \\
J(t) = \text{cst.}
\end{array} \right.$$
What does it require?

Inhomogeneous

$$(x, v)$$

$$(\theta, J)$$

Angle-Action coordinates

Relaxed

$$F = F(J, t)$$

Quasi-stationary states

Resonant

$$\Omega(J) = \frac{\partial H_0}{\partial J}$$

Fast/Slow timescale

Self-gravitating

$$\frac{1}{|\epsilon(\omega)|}$$

Linear response theory

Discrete & Perturbed

$$\frac{1}{N}$$

Finite-N effects
Collective effects

Self-gravitating amplification

**Gravitational polarisation** essential to

+ Cause dynamical instabilities
+ Induce *dynamical friction* and *mass segregation*
+ Accelerate/Slow down secular evolution
Gravitational polarisation essential to
+ Cause dynamical instabilities
+ Induce dynamical friction and mass segregation
+ Accelerate/Slow down secular evolution
Self-gravitating systems and Balescu-Lenard equation

Typical fate of a self-gravitating system

- **Initial conditions**
  - Relaxation: $\sim T_{\text{dyn}}$ (Baryon. dissip.)
  - Violent relaxation

- **External FP**
- **Internal BL**
- **Perturbations**
  - Phase mixing
  - Self-gravity

- **Quasi-stationary states**

- **Secular evolution**
  - $T_{\text{sec}} \gg T_{\text{dyn}}$

- **Linear instability**

- **Equilibrium**

- **Balescu-Lenard Equation**
What is the Balescu-Lenard Eq.?
Balescu-Lenard equation

The master equation for **self-induced orbital relaxation**

\[
\frac{\partial F(J, t)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial J} \cdot \left[ \sum_{k,k'} \int dJ' \frac{\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))}{|\epsilon_{kk'}(J, J', k \cdot \Omega(J))|^2} \right] \times \left( k \cdot \frac{\partial}{\partial J} - k' \cdot \frac{\partial}{\partial J'} \right) F(J, t) F(J', t)
\]

Some properties

- **\( F(J, t) \)**: Orbital distortion in **action space**
- **1/N**: Sourced by **finite-N effects**
- **\( \partial / \partial J \cdot \)**: Divergence of a **diffusion flux**
- **\( (k, k') \)**: Discrete **resonances**

- **\( \int dJ' \)**: Scan of **orbital space**
- **\( \delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J')) \)**: Resonance cond.
- **\( 1/|\epsilon_{kk'}(J, J', \omega)|^2 \)**: Dressed **couplings**
Resonant encounters

\[ \delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J')) \]

Collisions are **resonant, long-range, correlated**
Self-gravitating systems and Balescu-Lenard equation

Dressed resonant encounters

\[ \delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J')) \]

Collisions are resonant, long-range, correlated, and dressed

Fluctuations have a wake

\[ \delta \Phi \rightarrow | \varepsilon(\omega) | \]

Interactions between wakes

\[ D_{\text{diff}}(J) \rightarrow \frac{D_{\text{diff}}(J)}{| \varepsilon(\omega) |^2} \]
Non-local resonances

$$\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))$$

Non-local resonant couplings between dressed wakes
Diffusion is anisotropic

Generic diffusion equation

\[
\frac{\partial F(J, t)}{\partial t} = \frac{\partial}{\partial J} \cdot \left[ \sum_k k D_k(J) \cdot \frac{\partial F}{\partial J} \right]
\]

Two sources of anisotropies
Balescu-Lenard equation

The master equation for **self-induced orbital relaxation**

\[
\frac{\partial F(J, t)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial J} \cdot \left[ \sum_{k, k'} \int dJ' \ \delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J')) \left| \frac{\delta}{\delta J} \cdot \frac{\delta}{\delta J'} \right| F(J, t) F(J', t) \right]
\]

**Some properties**

- \( F(J, t) \): Orbital distortion in **action space**
- \( 1/N \): Sourced by **finite-N effects**
- \( \partial/\partial J \cdot \): Divergence of a **diffusion flux**
- \( (k, k') \): Discrete **resonances**

**Scan of orbital space**

\[
\int dJ' \delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J')) \left| \frac{\delta}{\delta J} \cdot \frac{\delta}{\delta J'} \right| F(J, t) F(J', t)
\]

**Resonance cond.**

\[
1/|\varepsilon_{kk'}(J, J', \omega)|^2
\]

**Dressed couplings**
Fokker-Planck equation

\[ \frac{\partial P(J, t)}{\partial t} = \frac{\partial}{\partial J} \left[ \sum_{k,k'} k \int dJ' \frac{\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))}{|\varepsilon_{kk'}(J, J', k \cdot \Omega(J))|^2} \right] \times \left( m_b k \cdot \frac{\partial}{\partial J} - m_t k' \cdot \frac{\partial}{\partial J'} \right) P(J, t) F_b(J', t) \]

Diffusion

\[ m_b k \cdot \frac{\partial}{\partial J} \]
Vanishes in the collisionless limit \( N \to +\infty \)
Sourced correlations in the potential fluctuations

Friction

\[ m_t k' \cdot \frac{\partial}{\partial J'} \]
Induces mass segregation
Sourced by the backreaction of the test particle on the bath
Where does the Balescu-Lenard Eq. come from?
Self-gravitating systems and Balescu-Lenard equation

Where does it come from?

Heyvaerts 10

Direct resolution of BBGKY
\[ \frac{\partial F}{\partial t} = \ldots ; \quad \frac{\partial G_2}{\partial t} = \ldots \]

Heyvaerts et al. 17

Fokker-Planck calculation
\[ \langle \frac{\Delta J}{\Delta t} \rangle ; \quad \langle \frac{\Delta J \otimes \Delta J}{\Delta t} \rangle \]

BBGKY and degenerate systems
\[ \forall J, \; n \cdot \Omega(J) = 0 \]

Difficulties

Diffusion in orbital space: \( F(J, t) \)

Accounting for collective effects: \( \frac{1}{|\epsilon_{kk}(J, J', \omega)|^2} \)

Timescale decoupling: \( \frac{\partial \langle F \rangle}{\partial t} \ll \frac{\partial \delta F}{\partial t} \)

Chavanis 12

Quasilinear Klimontovich equation
\[ \frac{\langle F \rangle}{\partial t} = \ldots ; \quad \frac{\partial \delta F}{\partial t} = \ldots \]

Functional approach
\[ i \int dt \; dw \; \lambda \left( \frac{\partial F}{\partial t} + \ldots \right) \]

Stochastic approach and Novikov theorem
\[ \frac{dJ}{dt} = \eta(\theta, J, t) \]
Self-gravitating systems and Balescu-Lenard equation

Where does it come from?

Direct resolution of BBGKY

\[
\frac{\partial F}{\partial t} = \ldots ; \quad \frac{\partial G_2}{\partial t} = \ldots
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Balescu-Lenard from BBGKY

N identical particles of mass \( m = \frac{M_{\text{tot}}}{N} \) in phase space \( \mathbf{w}_i = (\mathbf{x}_i, \mathbf{v}_i) \)

Total specific Hamiltonian

\[
H_N = \sum_{i=1}^{N} U_{\text{ext}}(\mathbf{w}_i) + \sum_{i<j}^{N} m U(\mathbf{w}_i, \mathbf{w}_j)
\]

System characterised by the N-body PDF \( P_N(\mathbf{w}_1, \ldots, \mathbf{w}_N, t) \)

Continuity equation in phase space

\[
\frac{\partial P_N}{\partial t} + \sum_i \frac{\partial}{\partial \mathbf{w}_i} \cdot \left( P_N \dot{\mathbf{w}}_i \right) = 0
\]

Exact Liouville equation

\[
\frac{\partial P_N}{\partial t} + [P_N, H_N]_N = 0
\]

3D self-gravitating systems

\[
U_{\text{ext}} = \frac{|\mathbf{v}|^2}{2} \\
U = -\frac{G}{|\mathbf{x} - \mathbf{x}'|}
\]
Reduced DFs

\[ F_n(w_1, \ldots, w_n, t) = m^n \frac{N!}{(N-n)!} \int dw_{n+1} \ldots dw_N P_N(w_1, \ldots, w_N, t) \]

BBGKY hierarchy

\[
\frac{\partial F_n}{\partial t} + [F_n, H_n]_n + \int dw_{n+1} [F_{n+1}, \delta H_{n+1}]_n = 0
\]

With

\[ H_n = \sum_{i=1}^{n} U_{\text{ext}}(w_i) + \sum_{i<j}^{N} m U(w_i, w_j) \]

n-body system

\[ \delta H_{n+1} = \sum_{i=1}^{n} U(w_i, w_{n+1}) \]

Interactions with (n+1)
Self-gravitating systems and Balescu-Lenard equation

**BBGKY at 1/N**

**Cluster representation** of the DFs

\[
\begin{align*}
F_2(w, w') &= F_1(w) F_1(w') + G_2(w, w') \\
F_3(w, w', w'') &= \ldots + G_3(w, w', w'')
\end{align*}
\]

\[\implies \begin{cases} G_2 \sim 1/N \\ G_3 \sim 1/N^2 \end{cases}\]

Truncation at order 1/N: 2 dynamical quantities

- \(F(w, t)\) 1-body DF
- \(G(w, w', t)\) 2-body correlation

**BBGKY - 1**

\[
\frac{\partial F}{\partial t} + [F, H_0]_w + \int dw' [G, U(w, w')]_w = 0
\]

**BBGKY - 2**

\[
\frac{\partial G}{\partial t} + [G, H_0]_w + \int dw'' G(w', w'') [F(w), U(w, w'')]_w \\
+ m [F(w) F(w'), U(w, w')]_w + (w \leftrightarrow w') = 0
\]
Self-gravitating systems and Balescu-Lenard equation

**BBGKY - 1**

\[
\frac{\partial F}{\partial t} + [F, H_0]_w + \int d\mathbf{w}' [G, U(\mathbf{w}, \mathbf{w}')]_w = 0
\]

- **Mean-field advection** \([F, H_0]_w\)
- **Collision term** \(\int d\mathbf{w}' [G, U(\mathbf{w}, \mathbf{w}')]_w\)

**BBGKY - 2**

\[
\frac{\partial G}{\partial t} + [G, H_0]_w + \int d\mathbf{w}'' G(\mathbf{w}', \mathbf{w}'') [F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')]_w + m [F(\mathbf{w}) F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')]_w = 0
\]

- **Mean-field advection** \([G, H_0]_w\)
- **Collective effects** \(\int d\mathbf{w}'' G(\mathbf{w}', \mathbf{w}'') [F(\mathbf{w}), U(\mathbf{w}, \mathbf{w}'')]_w\)
- **1-body DF sourcing** \([F(\mathbf{w}) F(\mathbf{w}'), U(\mathbf{w}, \mathbf{w}')]_w\)
How to solve BBGKY

Adiabatic approximation
i.e. evolution along quasi-stationary states

\[ F = F(J, t) \quad ; \quad H_0 = H_0(J, t) \quad \implies \quad [F_0(J), H_0(J)]_w = 0 \]

Mean-field equilibrium

Timescale separation

\[ \frac{\partial G}{\partial t} + [G, H_0]_w + (\ldots) = 0 \]

Collision operator

\[ \frac{\partial F}{\partial t} = - \int dw' [G, U(w, w')]_w \]

Bogoliubov’s Ansatz

\[ \frac{\partial G}{\partial t} = \text{BBGKY}_2[F = \text{cst}, G] \]

\[ \frac{\partial F}{\partial t} = \text{BBGKY}_1[F, G(t \to + \infty)] \]
The dynamics of correlations

Time evolution of the correlations

\[
\frac{\partial G(w, w')}{\partial t} + V_w(G) + V_{w'}(G) = S(w, w')
\]

Vlasov operator  Source term

Linearised Vlasov operator

\[
V_w(f(w)) = [f(w), H_0(w)]_w + \int dw' [f(w')F_0(w), U(w, w')]_w
\]

Mean field  Collective effects

Solved using Green’s functions

\[
G(w, w', t) = \int d\tilde{w} d\tilde{w}' \Green[w, w'|\tilde{w}, \tilde{w}', t] S(\tilde{w}, \tilde{w}', 0)
\]

Green’s function  Time-independent

Miracle: Vlasov operator acts independently on \((w, w')\)

\[
\Green[w, w'|\tilde{w}, \tilde{w}', t] = \Green[w|\tilde{w}, t] \Green[w'|\tilde{w}', t]
\]

Separability
Self-gravitating systems and Balescu-Lenard equation

Where does it come from?

Heyvaerts et al. 17

Direct resolution of BBGKY

\[
\frac{\partial F}{\partial t} = \ldots ; \quad \frac{\partial G_2}{\partial t} = \ldots
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Heyvaerts et al. 17

Fokker-Planck calculation

\[
\left\langle \frac{\Delta J}{\Delta t} \right\rangle ; \quad \left\langle \frac{\Delta J \otimes \Delta J}{\Delta t} \right\rangle
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Quasilinear Klimontovich equation

\[
\frac{\langle F \rangle}{\partial t} = \ldots ; \quad \frac{\partial \delta F}{\partial t} = \ldots
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Chavanis 12

Functional approach

\[
i \int dt \, d\mathbf{w} \, \lambda \left[ \frac{\partial F}{\partial t} + \ldots \right]
\]

Heyvaerts et al. 17

BBGKY and degenerate systems

\[
\forall J, \quad \mathbf{n} \cdot \Omega(J) = 0
\]

Heyvaerts et al. 17

Stochastic approach and Novikov theorem

\[
\frac{d J}{d t} = \eta(\theta, J, t)
\]

Chavanis 12

Difficulties

Diffusion in orbital space: \( F(J, t) \)

Accounting for collective effects: \( 1 / | \epsilon_{kk}(J, J', \omega) |^2 \)

Timescale decoupling: \( \partial \langle F \rangle / \partial t \ll \partial \delta F / \partial t \)
**Balescu-Lenard via Klimontovich**

Describing one realization in phase space \( \mathbf{w} = (\mathbf{x}, \mathbf{v}) \)

**Discrete DF**

\[
F_d(\mathbf{w}, t) = \sum_{i=1}^{N} m \delta_D(\mathbf{w} - \mathbf{w}_i(t))
\]

**Discrete Hamiltonian**

\[
H_d(\mathbf{w}, t) = U_{\text{ext}}(\mathbf{w}) + \int d\mathbf{w}' F_d(\mathbf{w}', t) U(\mathbf{w}, \mathbf{w}')
\]

**Continuity equation** in phase space

\[
\frac{\partial F_d}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot \left( F_d \mathbf{w} \right) = 0
\]

**Exact Klimontovich equation**

\[
\frac{\partial F_d}{\partial t} + [F_d, H_d] = 0
\]

3D gravitational systems

\[
U_{\text{ext}} = \frac{|\mathbf{v}|^2}{2}
\]

\[
U = -\frac{G}{|\mathbf{x} - \mathbf{x}'|}
\]
Solving Klimontovich

**Perturbative expansion**

\[
\begin{align*}
F_d &= F_0 + \delta F \quad \text{with} \quad \left\langle \delta F \right\rangle = 0, \\
H_d &= H_0 + \delta H \quad \text{with} \quad \left\langle \delta H \right\rangle = 0.
\end{align*}
\]

**Adiabatic approximation**

\[
\begin{align*}
F_0 &= F_0(\mathbf{J}, t), \\
H_0 &= H_0(\mathbf{J}, t).
\end{align*}
\]

**Quasi-linear evolution equations**

\[
\begin{align*}
\frac{\partial F_0}{\partial t} &= -\left\langle [\delta F, \delta H] \right\rangle, \\
\frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F_0, \delta H] &= 0
\end{align*}
\]

**Timescale separation**

\[
\begin{align*}
T_{\delta F} &\approx T_{\text{dyn}} \\
T_{F_0} &\approx (\sqrt{N})^2 \times T_{\delta F}
\end{align*}
\]
Self-gravitating systems and Balescu-Lenard equation

Dynamics of fluctuations

Fast evolution of perturbations (Linearised Klimontovich Eq.)

\[ \frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F_0, \delta H] = 0 \]

\[ [\delta F, H_0] \] Mean-field advection

\[ [F_0, \delta H] \] Collective effects

Self-consistent amplification

\[ \delta H = \delta H [\delta F] \]

Timescale separation

\[ \begin{cases} 
F_0(J) = \text{cst} \\
H_0(J) = \text{cst} 
\end{cases} \]

Phase Mixing
Self-gravitating systems and Balescu-Lenard equation

Solving for the fluctuations

Linear amplification

\[
\delta \hat{F}_k(J, \omega) = - \frac{\delta F_k(J, 0)}{i(\omega - k \cdot \Omega(J))} - \frac{k \cdot \partial F_0 / \partial J}{\omega - k \cdot \Omega(J)} \delta \hat{H}_k(J, \omega)
\]

with the self-consistency

\[
\delta H(w, t) = \int d w' \delta F(w', t) U(w, w')
\]

Generic form of a Fredholm equation

\[
\begin{bmatrix} \delta H(J) \end{bmatrix}_{\text{dressed}} = \begin{bmatrix} \delta H(J) \end{bmatrix}_{\text{bare}} + \int d J' M(J, J') \begin{bmatrix} \delta H(J') \end{bmatrix}_{\text{dressed}}
\]

Dressing of perturbations

\[
\begin{bmatrix} \delta H(\omega) \end{bmatrix}_{\text{dressed}} \approx \frac{\begin{bmatrix} \delta H(\omega) \end{bmatrix}_{\text{bare}}}{1 - M(\omega)} = \frac{\begin{bmatrix} \delta H(\omega) \end{bmatrix}_{\text{bare}}}{| \varepsilon(\omega) |}
\]

Amplification kernel
Self-gravitating systems and Balescu-Lenard equation

**Basis method** \((\psi^{(p)}(w), \rho^{(p)}(w))\)

\[
\begin{aligned}
\psi^{(p)}(w) &= \int dw' U(w, w') \rho^{(p)}(w'), \\
\int dw \psi^{(p)}(w) \rho^{(q)*}(w) &= -\delta_{pq}.
\end{aligned}
\]

"Separable" pairwise interaction

\[
U(w, w') = -\sum_p \psi^{(p)}(w) \psi^{(p)*}(w')
\]

**Plasmas**

\[
U(x, x') = \frac{1}{|x - x'|}
\]

\[
\iint \frac{dk}{|k|^2} e^{ik \cdot x} e^{-ik \cdot x'}
\]

**Galaxies**

\[
\Delta \Phi = 4\pi G\rho
\]

Poisson equation
Self-gravitating systems and Balescu-Lenard equation

Linear response theory

\[
[\delta H(\omega)]_{\text{dressed}} = \frac{[\delta H(\omega)]_{\text{bare}}}{|\varepsilon(\omega)|}
\]

\[
\varepsilon_{pq}(\omega) = 1 - \sum_k \int dJ \frac{k \cdot \partial F_0 / \partial J}{\omega - k \cdot \Omega(J)} \psi^*_k(J) \psi_q(J)
\]

Dielectric function

Two limits

\[
\varepsilon_{pq}(\omega) \approx 0 \quad \text{Cold regime}
\]

\[
\varepsilon_{pq}(\omega) \approx 1 \quad \text{Hot regime}
\]

Some properties

\[
\sum_k \int dJ
\]

Sum over resonances

Scan over orbital space

Resonant int.

Long-range int.
Self-gravitating systems and Balescu-Lenard equation

Dielectric function

\[
\begin{align*}
\text{Im}[\omega] & \quad \text{Re}[\omega] \\
\end{align*}
\]

1

\[|\epsilon(\omega)| \]

\(\Omega_p\)

Damped mode

Linearly stable system

Susceptibility

\[
\frac{1}{|\epsilon(\Omega_p)|} \gg 1
\]

Thermalisation

\[
[\delta H(t)]_{\text{trans.}} \simeq e^{-\eta_p t}
\]
Dressed long-term diffusion

**Secular** evolution equation

\[
\frac{\partial F_0}{\partial t} = - \langle [\delta F, \delta H] \rangle
\]

**Dressing** comes twice

\[
\left[ \delta H \right]_{\text{dressed}} = \frac{\left[ \delta H \right]_{\text{bare}}}{| \varepsilon(\omega) |}
\]

**Bare** Poisson shot noise

\[
| \delta H |_{\text{bare}} \simeq \frac{1}{\sqrt{N}}
\]

**Relaxation time**

\[
\frac{\partial F_0}{\partial t} \simeq \frac{| \delta H |^2_{\text{bare}}}{| \varepsilon(\omega) |^2}
\]

Collective effects can **drastically accelerate** orbital heating, in particular on **large scales**
The master equation for **self-induced orbital relaxation**

$$\frac{\partial F(J, t)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial J} \cdot \left[ \sum_{k,k'} \int dJ' \delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J')) \frac{1}{|\epsilon_{kk'}(J, J', k \cdot \Omega(J))|^2} \times \left( k \cdot \frac{\partial}{\partial J} - k' \cdot \frac{\partial}{\partial J'} \right) F(J, t) F(J', t) \right]$$

**Some properties**

- $F(J, t)$: Orbital distortion in **action space**
- $1/N$: Sourced by **finite-N effects**
- $\partial / \partial J \cdot$: Divergence of a **diffusion flux**
- $(k, k')$: Discrete **resonances**

**Scan of orbital space**

$$\int dJ'$$

**Resonance cond.**

$$\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))$$

**Dressed couplings**

$$\frac{1}{|\epsilon_{kk'}(J, J', \omega)|^2}$$
Self-gravitating systems and Balescu-Lenard equation

Plasmas

Orbital coordinates

\((x, v)\)

Basis decomposition

\(U(x, x') \propto \int \frac{dk}{k^2} e^{i k \cdot (x-x')}\)

Dielectric function

\[1 - \frac{1}{k^2} \int dv \frac{k \cdot \partial F/\partial v}{\omega - k \cdot v}\]

Resonance condition

\(\delta_D(k \cdot (v - v'))\)

Galaxies

Orbital coordinates

\((\theta, J)\)

Basis decomposition

\(U(w, w') = -\sum_p \psi^{(p)}(w) \psi^{(p)*}(w')\)

Dielectric function

\[\delta_{pq} - \sum_k \int dJ \frac{k \cdot \partial F/\partial J}{\omega - k \cdot \Omega(J)} \psi_k^{(p)*}(J) \psi_k^{(q)}(J)\]

Resonance condition

\(\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))\)
Does the Balescu-Lenard Eq. work?
Long-range interacting systems are ubiquitous

### Homogeneous systems

\[ \psi = -\frac{1}{|x - x'|} \]

d=3, homogeneous

### Hamiltonian Mean Field Model

\[ \psi = -\cos(\theta - \theta') \]

d=1, inhomogeneous

### Vector Resonant Relaxation

\[ \psi = -V(s \cdot s') \]

d=1, inhomogeneous, degenerate

### 2D hydrodynamics

\[ \psi = -\ln(|x - x'|) \]

d=2, inhomogeneous

### Self-gravitating discs

\[ \psi = -\frac{1}{|x - x'|} \]

d=2, inhomogeneous

### Scalar Resonant Relaxation

\[ \psi = -\int \frac{d\theta d\theta'}{|x - x'|} \]

d=2, inhomogeneous, degenerate
The diversity of long-range interacting systems

<table>
<thead>
<tr>
<th>Small dimension</th>
<th>Large dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 1$</td>
<td>$d = 2$</td>
</tr>
<tr>
<td>Galactic Nuclei</td>
<td>Globular clusters</td>
</tr>
<tr>
<td>Galactic discs</td>
<td>Galactic discs</td>
</tr>
<tr>
<td>Dark matter halo</td>
<td>Dark matter halo</td>
</tr>
<tr>
<td>$\frac{1}{</td>
<td>\varepsilon(\omega)</td>
</tr>
<tr>
<td>Non-degenerate</td>
<td>Degenerate</td>
</tr>
<tr>
<td>No global resonance</td>
<td>$\forall J, n \cdot \Omega(J) = 0$</td>
</tr>
</tbody>
</table>

Homogeneous $(x, v)$

- Hot
- $\frac{1}{|\varepsilon(\omega)|} \simeq 1$

Inhomogeneous $(\theta, J)$

- Cold
- $\frac{1}{|\varepsilon(\omega)|} \gg 1$

Non-degenerate

Galaxies

Degenerate
**Self-gravitating systems and Balescu-Lenard equation**

### Balescu-Lenard: A numerical nightmare

\[ \frac{\partial F(J, t)}{\partial t} = - \frac{\partial}{\partial J} \cdot F(J, t) \]

**Balescu-Lenard equation**

\[ F(J, t) = \sum_{k,k'} k \int dJ' \frac{\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))}{|\varepsilon_{kk}(J, J', k \cdot \Omega(J))|^2} \times \left(k' \cdot \frac{\partial}{\partial J'} - k \cdot \frac{\partial}{\partial J}\right) F(J) F(J') \]

**Diffusion flux**

\[ \frac{1}{\varepsilon_{kk}(J, J', \omega)} = \sum_{p,q} \psi^{(p)}_k(J) E_{pq}^{-1}(\omega) \psi^{(q)*}_k(J') \]

**Dressed susceptibility coefficients**

\[ E_{pq}(\omega) = \delta_{pq} - M_{pq}(\omega) \]

**Dielectric matrix**

\[ M_{pq}(\omega) = \sum_k \int dJ \frac{k \cdot \partial F/\partial J}{\omega - k \cdot \Omega(J)} \psi^{(p)*}_k(J) \psi^{(q)}_k(J) \]

**Response matrix**

\[ \psi^{(p)}_k(J) = \int \frac{d\theta}{(2\pi)^d} \psi^{(p)}(x[\theta, J]) e^{-ik \cdot \theta} \]

**Basis elements**
Self-gravitating systems and Balescu-Lenard equation

\[ F(J, t) = \sum k \int dJ' \frac{\delta(D(k \cdot \Omega(J) - k' \cdot \Omega(J')))}{|\varepsilon_{kk}(J, J', k \cdot \Omega(J))|^2} \]

\[ \times \left( k' \frac{\partial}{\partial J'} - k \frac{\partial}{\partial J} \right) F(J) F(J') \]

Diffusion flux

\[ \frac{1}{\varepsilon_{kk}(J, J', \omega)} = \sum_{p, q} \psi_k^{(p)}(J) E^{-1}_{pq}(\omega) \psi_k^{(q)*}(J') \]

Dressed susceptibility coefficients

\[ E_{pq}(\omega) = \delta_{pq} - M_{pq}(\omega) \]

Dielectric matrix

\[ M_{pq}(\omega) = \sum_k \int dJ \frac{k \cdot \partial F/\partial J}{\omega - k \cdot \Omega(J)} \psi_k^{(p)*}(J) \psi_k^{(q)}(J) \]

Response matrix

\[ \psi_k^{(p)}(J) = \int \frac{d\theta}{(2\pi)^d} \psi^{(p)}(x[\theta, J]) e^{-ik \cdot \theta} \]

Basis elements

With also:

+ Integral over \( d\theta \)
+ (Double) integral over \( dJ \)
+ (Triple) sum over \( k \)
+ (Double) sum over \( (p, q) \)
+ Matrix inversion
+ Resonant denominator
+ Resonance condition

A numerical nightmare
Does it work?

Galactic discs

Globular clusters

Galactic nuclei
Self-gravitating systems and Balescu-Lenard equation

Does it work?

**Galactic discs**

\[
\frac{1}{|\varepsilon(\omega)|} \gg 1
\]

Dynamically cold system

**Globular clusters**

\((k, k') \in [1, +\infty]\)

Large number of resonances

**Galactic nuclei**

\[
U(w, w') \leftrightarrow \bar{U} = \int \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} U
\]

Orbit-averaged interactions
Self-gravitating systems and Balescu-Lenard equation

Does it work?

Galactic discs

\[ \frac{1}{|\epsilon(\omega)|} \gg 1 \]

Dynamically cold system

Globular clusters

\((k, k') \in [1, +\infty]\)

Large number of resonances

Galactic nuclei

\[ U(w, w') \mapsto \bar{U} = \int \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} U \]

Orbit-averaged interactions
Galactic discs

How do stars diffuse in galactic discs?
+ Galactic archeology
+ Formation of spiral arms/bars
+ Local velocity anisotropies
+ Disc thickening
+ Stellar streams

Swing amplification in cold discs

Sub-structures in action space, as observed by GAIA

\[ \frac{1}{|\varepsilon(\omega)|} \approx 30 \]

Collective effects essential

Trick et al., 2018

Toomre, 1981
Prediction for the diffusion

Diffusion flux in action space

\[
\frac{\partial F(J, t)}{\partial t} = - \frac{\partial}{\partial J} \cdot F(J, t)
\]

Spontaneous formation of anisotropic sub-structures in action space

It works!
Self-gravitating systems and Balescu-Lenard equation

Galactic discs

\[
\frac{1}{|\epsilon(\omega)|} \gg 1
\]

Dynamically cold system

Globular clusters

\[(k, k') \in [1, +\infty]\]

Large number of resonances

Galactic nuclei

\[
U(w, w') \leftrightarrow \overline{U} = \int \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} U
\]

Orbit-averaged interactions

Does it work?
Galactic centers

What is the diet of a supermassive black hole?

Stellar diffusion in galactic centers
+ Origin and structure of SgrA*
+ Relaxation in eccentricity, orientation

Sources of gravitational waves
+ BHs-binary mergers
+ TDE, EMRIs

What is the long-term dynamics of stars in these very dense systems?

S-Cluster of SgrA*
Densest stellar system of the galaxy
Dynamics dominated by the central black hole

Tidal Disruption Event
Extreme Mass Ratio Inspiral

Keck/UCLA Galactic Center Group

J. Guillenchenon

C. Sopuerta
Galactic centers

Domination by the **central BH**

\[ \forall J, \, n \cdot \Omega_{Kep}(J) = 0 \]

Degenerate dynamics

**Orbit-average**

Stars → Wires

Dynamics of the **wires**

**In-plane precessions**

\[ \Omega_{\text{prec}} = \Omega_\star + \Omega_{\text{rel}} \]

**Relaxation** of wires’ eccentricity via **Balescu-Lenard**
Galactic centers

Jitters of the **wires**

- Second orbit-average
  - **Wires** → **Annuli**

Dynamics of **annuli**

- Out-of-plane precessions
  - $\Omega_{\text{out}} = \Omega_\star + \Omega_{\text{spin}}^{\text{rel}}$

- Relaxation of wires’ orientation via **Balescu-Lenard**
Resonant Relaxation in Galactic nuclei

Relaxation of eccentricities

Relaxation of orientations

It works!
Self-gravitating systems and Balescu-Lenard equation

Does it work?

Galactic discs

\[ \frac{1}{| \epsilon(\omega) |} \gg 1 \]

Dynamically cold system

Galactic nuclei

\[ U(w, w') \mapsto \overline{U} = \int \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} U \]

Orbit-averaged interactions

Globular clusters

\[(k, k') \in [1, + \infty] \]

Large number of resonances
Globular clusters

M80, an example of globular clusters
Dense, spherical stellar systems, without a central BH

What is the very long-term evolution of globular clusters?
+ Orbital heating
+ Core collapse
+ Velocity anisotropies
+ Relaxation of orientations
+ Mass segregation

What is the long-term dynamics of globular clusters?

\[
\begin{align*}
R_{\text{sys}} & \approx 1\text{pc} \\
N & \approx 10^5 \\
T_{\text{life}} & \approx 10^{10}\text{yr} \\
T_{\text{dyn}} & \approx 10^5\text{yr} \\
T_{\text{relax}} & \approx 10^{10}\text{yr}
\end{align*}
\]
**Balescu-Lenard prediction**

**Diffusion flux in action space**

\[
\frac{\partial F(J, t)}{\partial t} = - \frac{\partial}{\partial J} \cdot F(J, t)
\]

Collective effects are essential

Balescu-Lenard better than Chandrasekhar, but still very unsatisfactory
What’s next?
What’s next?

Resonances

\[ k, k' \to +\infty \]

\[ \Omega(J) = \text{cst} \]

Kinetic blockings

\[ d = 1 \quad \text{and} \quad \frac{1}{N^2} \]

Deviations

\[ \frac{\partial F_d}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_d \rangle}{\partial t} \]

Integrability

\[ \Phi(x, t) \neq \Phi(J, t) \]
Self-gravitating systems and Balescu-Lenard equation

What’s next?

Resonances

\[ k, k' \to +\infty \]
\[ \Omega(J) = \text{cst} \]

Kinetic blockings

\[ d = 1 \quad \text{and} \quad \frac{1}{N^2} \]

Deviations

\[ \frac{\partial F_d}{\partial t} \quad \text{VS} \quad \frac{\partial \langle F_d \rangle}{\partial t} \]

Integrability

\[ \Phi(x, t) \neq \Phi(J, t) \]
Self-gravitating systems and Balescu-Lenard equation

(Non)-resonant relaxation

What about high-order resonances?

\[
\frac{\partial F(J, t)}{\partial t} = \frac{\partial}{\partial J} \cdot \left[ \sum_{k, k' \in \mathbb{Z}^3} \left( \ldots \right) \right]
\]

Resonant Relaxation

| k |, | k' | \approx 1

Long-range resonances

Where is the Coulomb logarithm?

Non-Resonant Relaxation

| k |, | k' | \gg 1

Local deflections

\[ \ln \Lambda = \ln\left(\frac{k_{\text{min}}}{k_{\text{max}}}\right) \]
Fundamental degeneracies

Dynamics in degenerate frequency profiles

\[ \delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J')) \]

Resonance condition

∀ J, \( \Omega(J) = 0 \)

∀ J, \( \Omega(J) = \Omega_0 \)

ψ = \( \hat{L} \cdot \hat{L}' \)

ψ = \[ -\int \frac{d\theta d\theta'}{|x-x'|} \]

How does relaxation occur in degenerate systems?
What’s next?

Resonances

\( k, k' \to +\infty \)
\[ \Omega(J) = \text{cst} \]

Deviations

\[ \frac{\partial F_d}{\partial t} \text{ vs } \frac{\partial \langle F_d \rangle}{\partial t} \]

Kinetic blockings

\[ d = 1 \quad \text{and} \quad \frac{1}{N^2} \]

Integrability

\[ \Phi(x, t) \neq \Phi(J, t) \]
Self-gravitating systems and Balescu-Lenard equation

Kinetic blockings

Generic **Balescu-Lenard** equation

\[
\frac{\partial F(J, t)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial J} \cdot \left[ \sum_{k,k'} k \int dJ' \frac{\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))}{|\varepsilon_{kk'}(J, J', k \cdot \Omega(J))|^2} \times \left( k \cdot \frac{\partial}{\partial J} - k' \cdot \frac{\partial}{\partial J'} \right) F(J, t) F(J', t) \right]
\]

What happens in **1D systems**?

\[
\begin{align*}
    k &= k' = k \\
    J &= J' = J
\end{align*}
\]

**Conspiracy** for 2-body effects in 1D

\[
\begin{align*}
    v_1 + v_2 &= \text{cst} \\
    v_1^2 + v_2^2 &= \text{cst}
\end{align*}
\]

No relaxation!

\[
\frac{\partial F(J, t)}{\partial t} = \frac{1}{N} \times 0
\]

\[
\psi = -\cos(\theta - \theta')
\]

Homogeneous HMF model
**Kinetic theory at order** \(1/N^2\)

**\(1/N^2\) kinetic equation**

\[
\frac{\partial F(v_1)}{\partial t} = \frac{1}{N^2} \frac{\partial}{\partial v_1} \left[ \mathcal{P} \int \frac{d v_1}{(v_1 - v_2)^4} \int d v_3 \right]
\]

\[
\times \left\{ \delta_D(2v_1 - v_2 - v_3) \left( 2 \frac{\partial}{\partial v_1} - \frac{\partial}{\partial v_2} - \frac{\partial}{\partial v_3} \right) F(v_1) F(v_2) F(v_3) \right\}
\]

\(\left. \frac{d}{d} \right|_{(v_1 \leftrightarrow v_2)} \}

+ How do collective effects contribute?
+ How do higher-order resonances contribute?
+ How do frequency profiles contribute?
+ What is the structure of kinetic theories at higher order \(1/N^s\)?
Self-gravitating systems and Balescu-Lenard equation

What’s next?

Resonances

\[ k, k' \rightarrow + \infty \]

\[ \Omega(J) = \text{cst} \]

Kinetic blockings

\[ d = 1 \quad \text{and} \quad \frac{1}{N^2} \]

Deviations

\[ \frac{\partial F_d}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_d \rangle}{\partial t} \]

Integrability

\[ \Phi(x, t) \neq \Phi(J, t) \]
Faking the dynamics

Kinetic theory predicts the **ensemble average** dynamics

\[ F_d \]

Realisations

Time

Ensemble Average

\[ \langle F_d \rangle \]
Self-gravitating systems and Balescu-Lenard equation

Faking the dynamics

\[ S(t) \]

One realisation vs. the **mean kinetic prediction**
Self-gravitating systems and Balescu-Lenard equation

Faking the dynamics

Probability of a given realisation?

\[ \mathbb{P}(F_d(t) = F_0(t)) \]

maximal for

\[ \mathbb{P}(F_d(t) = \langle F_d(t) \rangle) \]

Can one **fake** realisations?

\[ \frac{\partial F_d}{\partial t} = BL[F_d(t)] + \eta[F_d(t)] \]

with the noise

\[ \langle \eta[F_d] \eta[F_d] \rangle = ?? \]

What is the statistics of **(large)** deviations?
What’s next?

Resonances

\[ k, k' \to +\infty \]

\[ \Omega(J) = \text{cst} \]

Kinetic blockings

\[ d = 1 \quad \text{and} \quad \frac{1}{N^2} \]

Deviations

\[ \frac{\partial F_d}{\partial t} \quad \text{vs} \quad \frac{\partial \langle F_d \rangle}{\partial t} \]

Integrability

\[ \Phi(x, t) \neq \Phi(J, t) \]
Going beyond isolated, integrable, resonant

Systems are not always isolated

\[
\begin{aligned}
N &= N(t) \\
[\delta H(t)]_{\text{tot}} &= [\delta H(t)]_{\text{Poisson}} + [\delta H(t)]_{\text{ext}}
\end{aligned}
\]

Systems are not always integrable

\[
\left[ \frac{d\mathbf{J}}{dt} \right]_{\text{tot}} = \left[ \frac{d\mathbf{J}}{dt} \right]_{\text{resonant}} + \left[ \frac{d\mathbf{J}}{dt} \right]_{\text{chaotic}}
\]

Systems are not always "nicely" resonant

\[
\Omega(J) = \left( \Omega_1(J), \epsilon \Omega_2(J) \right)
\]

Structure formation
Open clusters
Collisionless relaxation

Thickened discs
Barred galaxies
Flattened halos

Mean-motion resonances
Eviction resonances
Precession resonances
Conclusions
Kinetic theory of self-gravitating systems

Long-range interacting systems are ubiquitous

Inhomogeneous

\[(x, v) \downarrow (\theta, J)\]

Self-gravitating

\[
\frac{1}{|\varepsilon(\omega)|}
\]

Resonant

\[k \cdot \Omega(J)\]

Master equation for dressed resonant relaxation

\[
\frac{\partial F(J, t)}{\partial t} = \frac{1}{N} \frac{\partial}{\partial J} \cdot \left\{ \sum_{k,k'} k \int dJ' \frac{\delta_D(k \cdot \Omega(J) - k' \cdot \Omega(J'))}{|\varepsilon_{kk}(J, J', k \cdot \Omega(J))|^2} \right. \\
\times \left. \left( k \cdot \frac{\partial}{\partial J} - k' \cdot \frac{\partial}{\partial J'} \right) F(J, t) F(J', t) \right\}
\]

Framework mature enough to be confronted to observations