Ridge formation in stellar discs from Poisson shot noise

Jean-Baptiste Fouvry, IAP

Oxford
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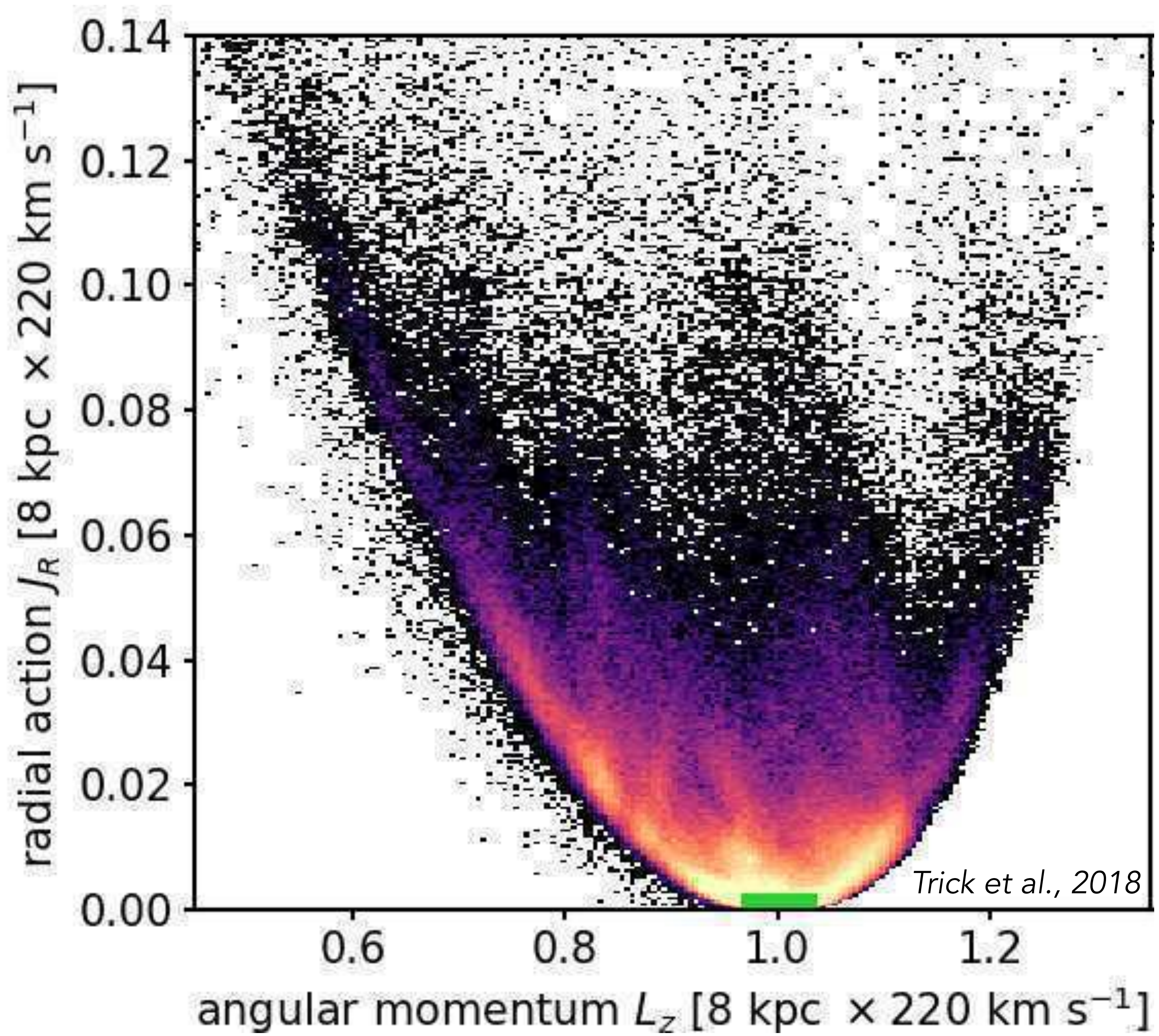
In collaboration with C. Pichon, J. Binney, J. Magorrian, P.-H. Chavanis, S. De Rijcke

(Resonant) ridges are everywhere



What is the origin of these orbital sub-structures?

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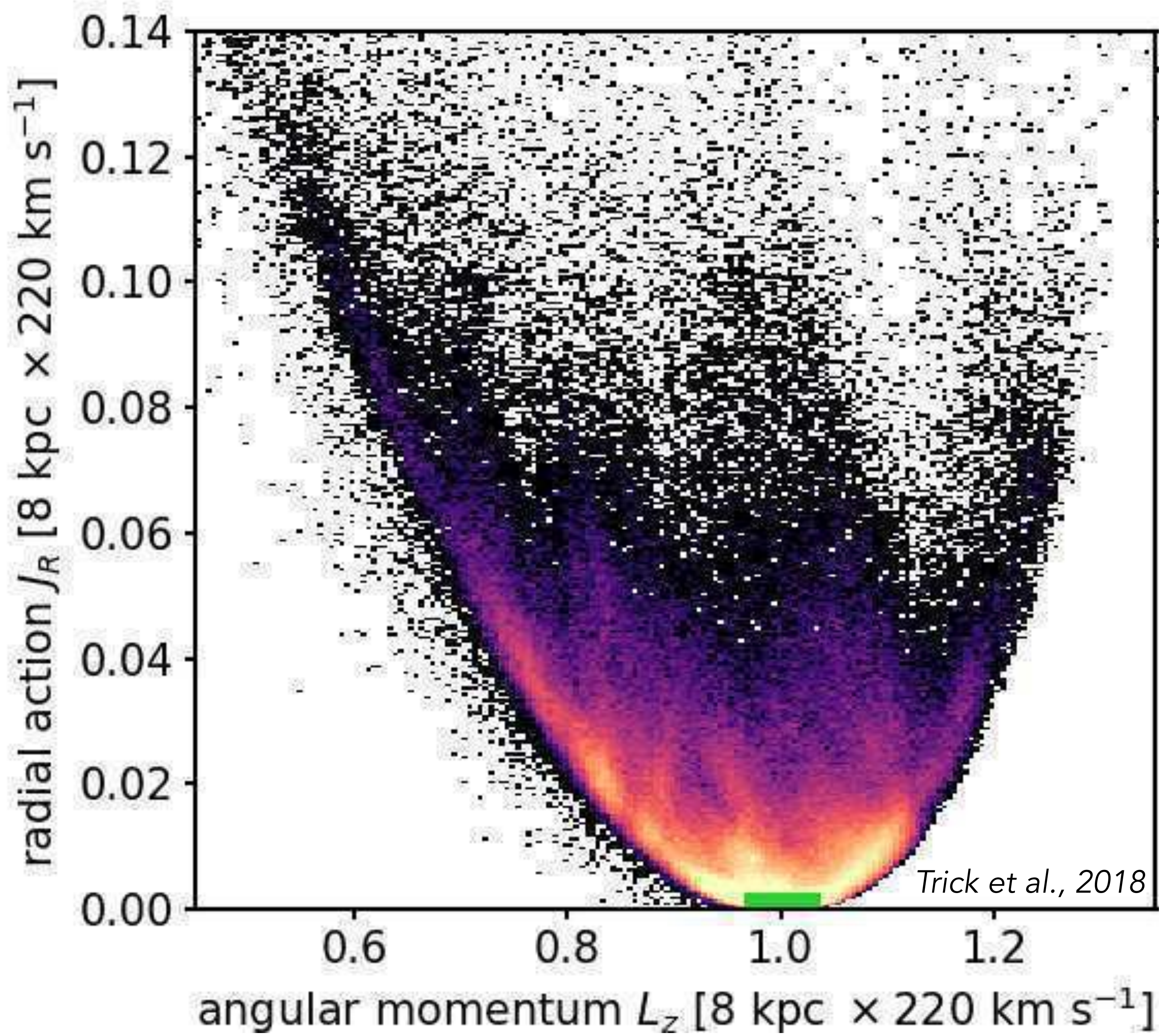
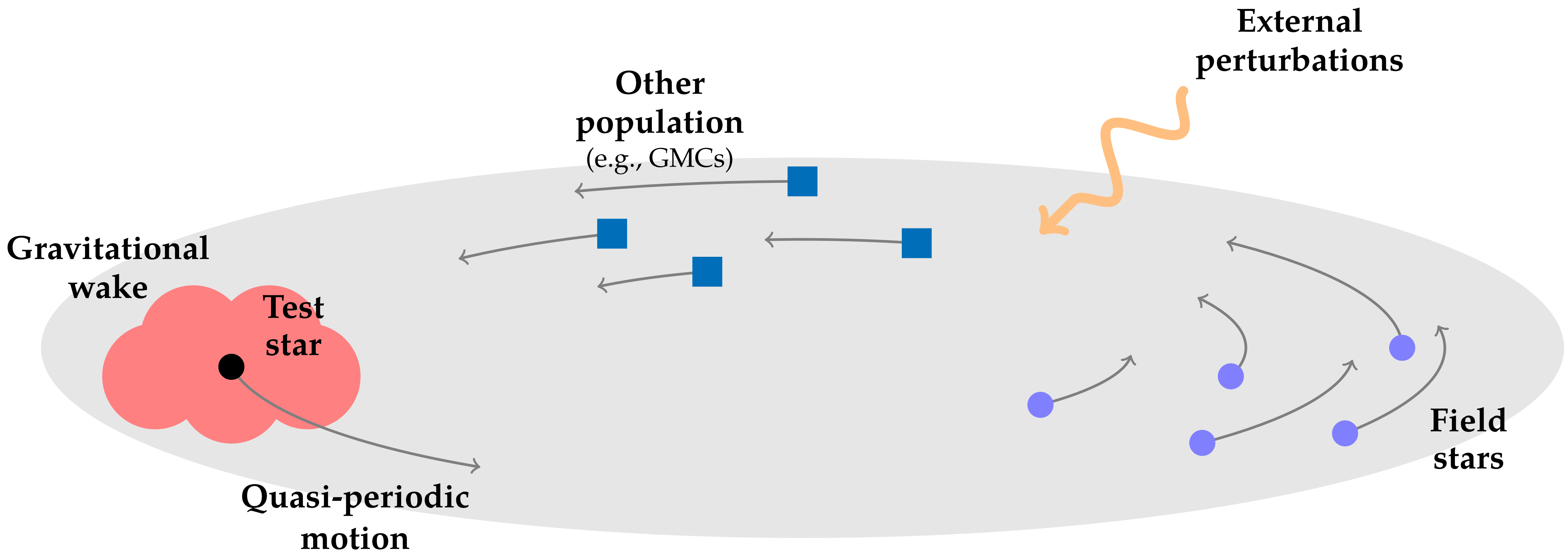


Dynamical vs. Secular	External vs. Internal	Resonant vs. Stochastic

Long-term dynamics of stellar discs



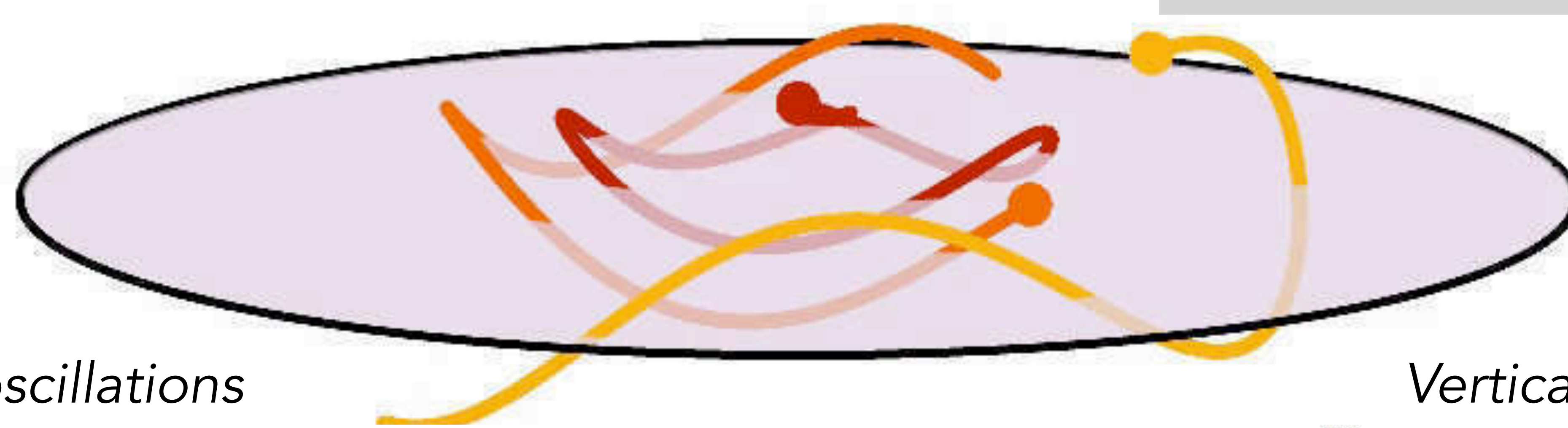
Galaxies are:

+ **Inhomogeneous** (complex trajectories) — *Angle-action coordinates*
+ **Resonant** (orbital frequencies) — *Fast timescale vs. cosmic timescale*

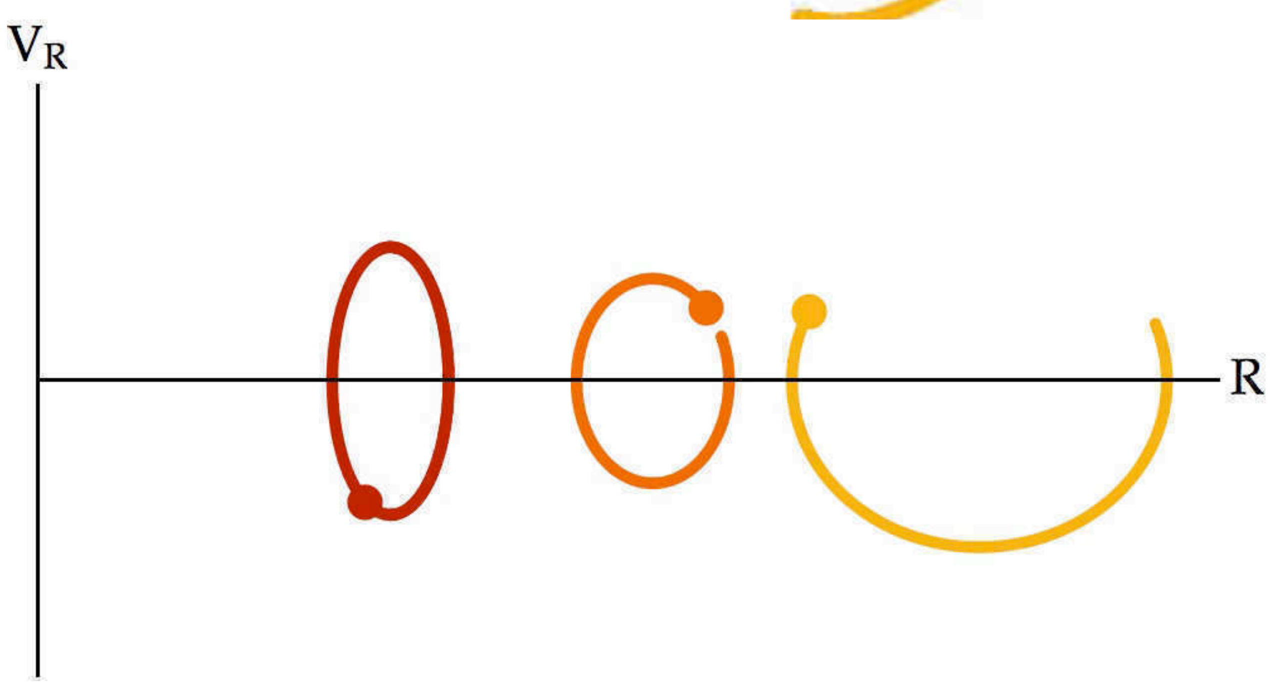
How to label orbits in a disc

Integrable orbits $\psi_0 = \psi_0(R, z)$

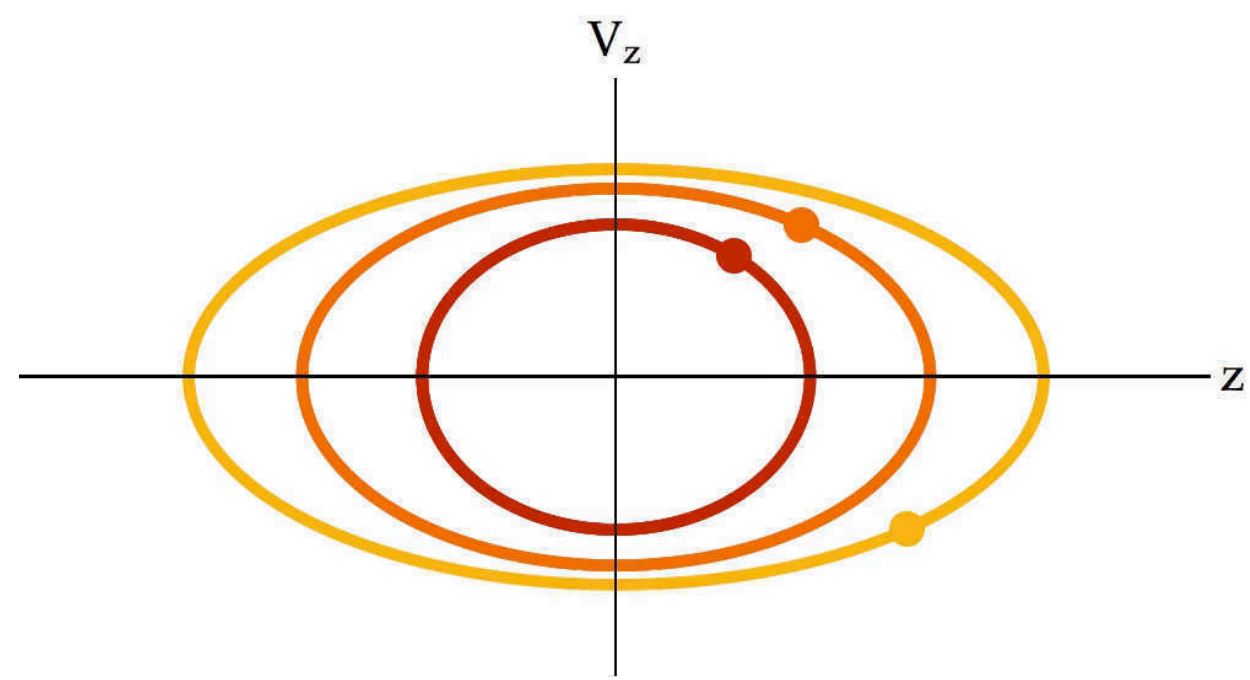
$$\begin{cases} \boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + t\,\boldsymbol{\Omega}(\mathbf{J}) \\ \mathbf{J}(t) = \mathrm{cst.} \end{cases}$$



Radial oscillations



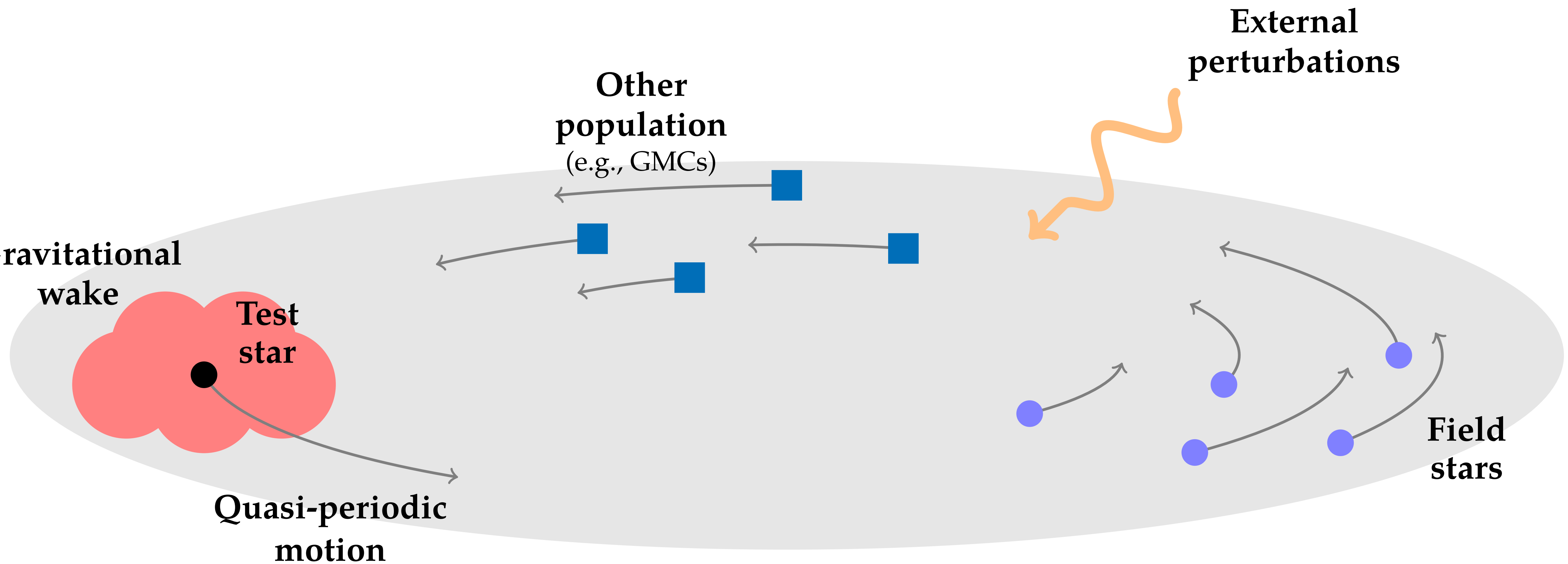
Vertical oscillations



Actions $\mathbf{J} = \left(J_\phi\,, J_r\,, J_z\right)$

Frequencies $\boldsymbol{\Omega} = \left(\Omega_\phi, \Omega_r, \Omega_z\right)$

Long-term dynamics of stellar discs

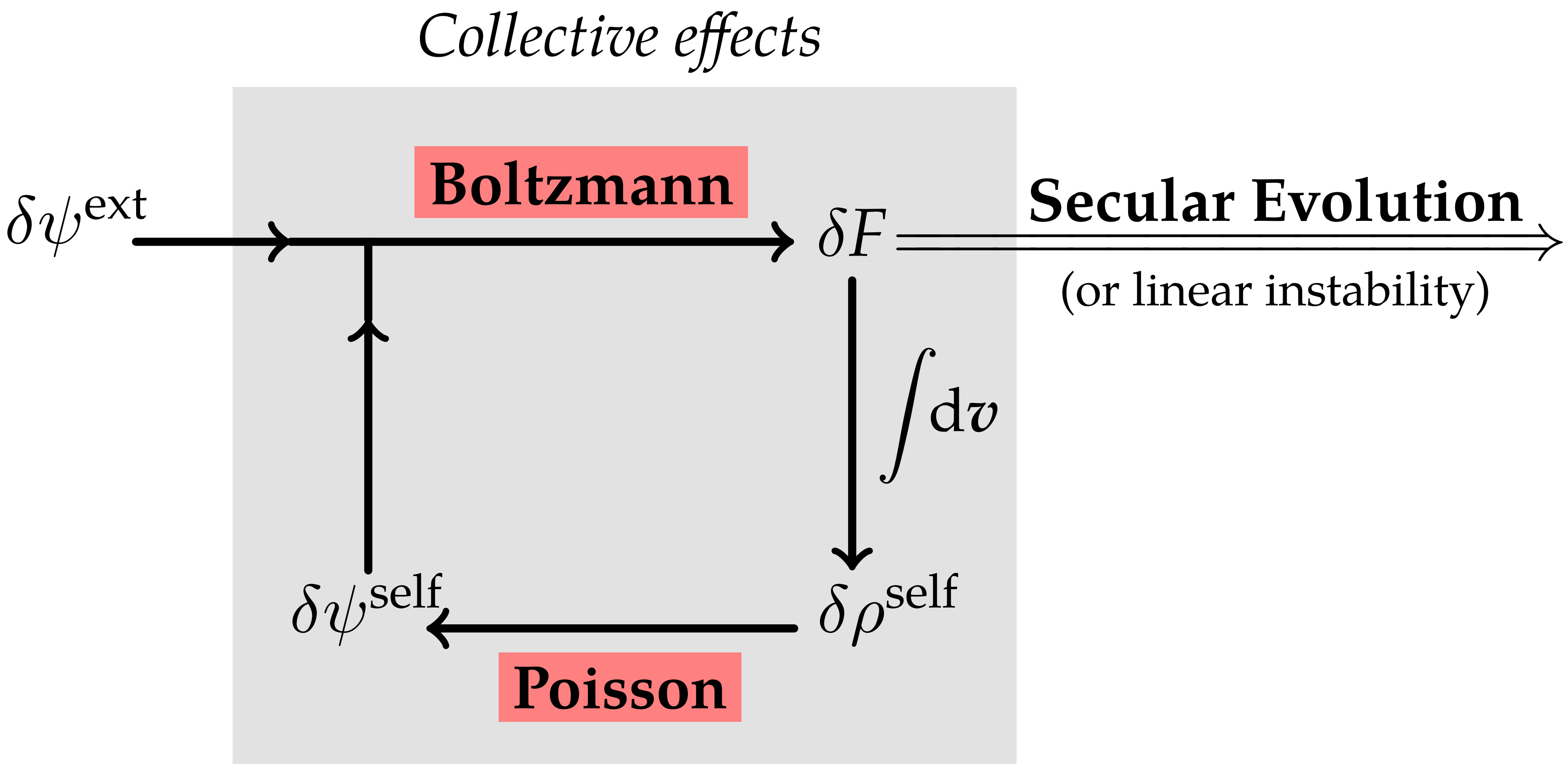


Galaxies are:

- + **Inhomogeneous** (complex trajectories) — *Angle-action coordinates*
- + **Resonant** (orbital frequencies) — *Fast timescale vs. cosmic timescale*
- + **Self-gravitating** (amplification of perturbations) — *Linear response theory*

Self-gravity

Self-gravitating amplification

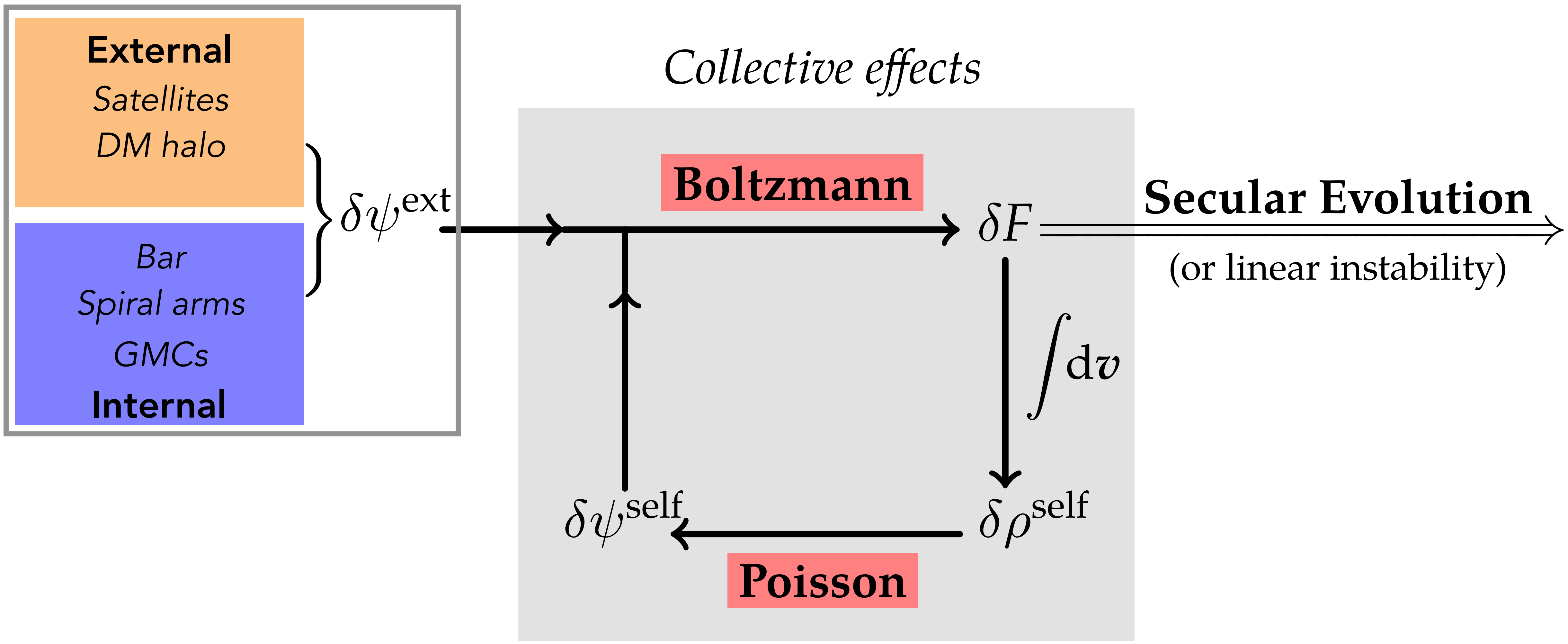


Gravitational polarisation essential to

+ *Cause dynamical instabilities*
+ *Induce **dynamical friction** and **mass segregation***
+ ***Accelerate/Slow down** secular evolution*

Collective effects and perturbations

Self-gravitating amplification



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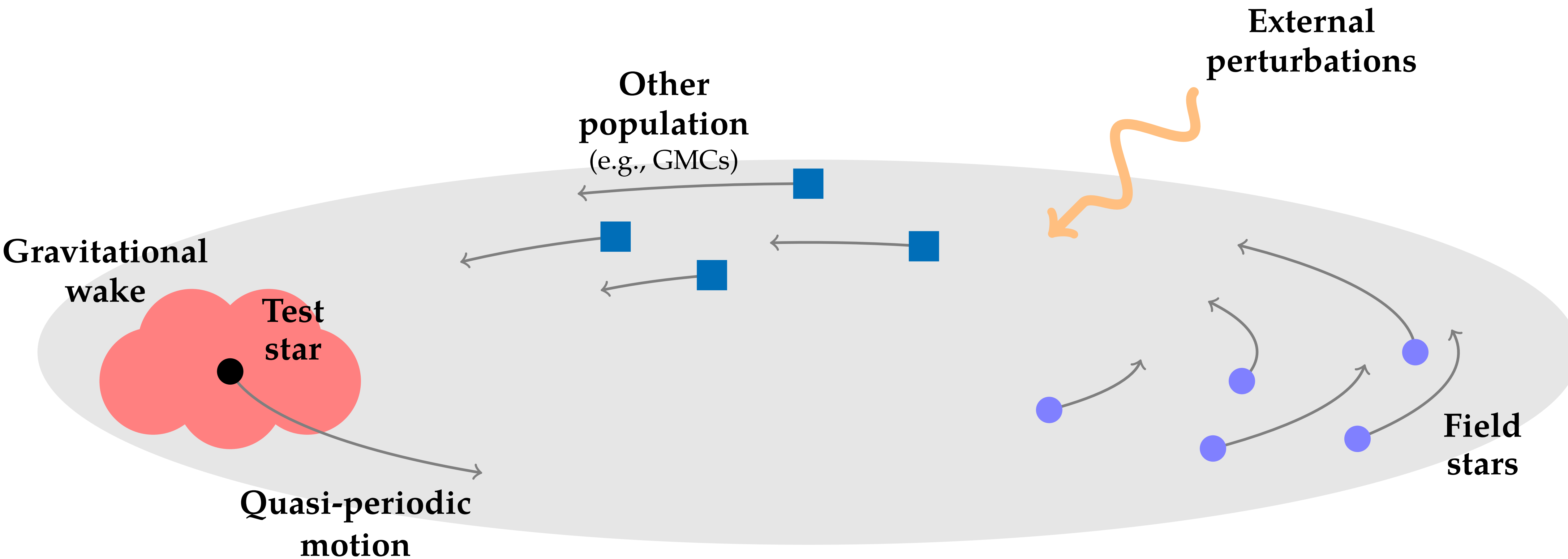
Long-term dynamics of stellar discs



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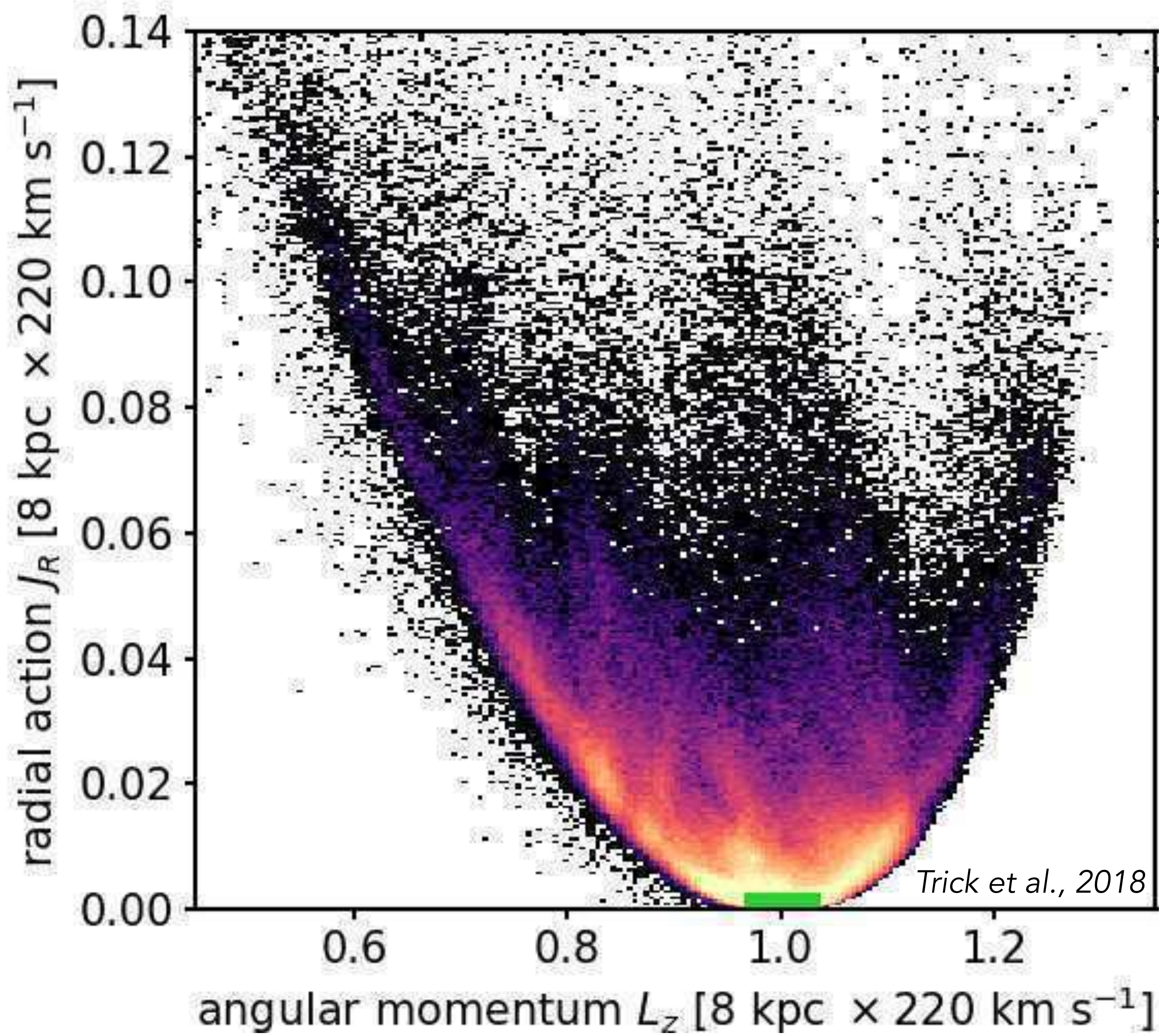

What is the origin of these orbital sub-structures?

How to describe the dynamics of a disc?

Perturbative expansion

$$F_{\mathrm{tot}}(\boldsymbol{\theta}, \mathbf{J}, t) = F_0(\mathbf{J}) + \delta F(\boldsymbol{\theta}, \mathbf{J}, t)$$

$$H_{\mathrm{tot}}(\boldsymbol{\theta}, \mathbf{J}, t) = H_0(\mathbf{J}) + \delta\psi^{\mathrm{ext}}(\boldsymbol{\theta}, \mathbf{J}, t) + \delta\psi^{\mathrm{self}}(\boldsymbol{\theta}, \mathbf{J}, t)$$

Self-consistency

$$\delta\psi^{\mathrm{self}}(\mathbf{x}) = -G \int \mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{v}' \frac{\delta F(\mathbf{x}', \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|}$$

Collisionless dynamics (**Vlasov** equation)

$$\frac{\partial F_{\mathrm{tot}}}{\partial t} + \left[F_{\mathrm{tot}}, H_{\mathrm{tot}}\right] = 0$$

Mean-field equilibrium modelling

Searching for an **equilibrium** solution

$$\left[F_0(\mathbf{J})\ ,\ H_0(\mathbf{J})\right] = 0 \implies \frac{\partial F_0(\mathbf{J},t)}{\partial t} = 0$$

Self-consistent solution (i.e. Jeans modelling)

$\Phi_0(R,z)$ → *Angle-action mapping* → $(\mathbf{x},\mathbf{v}) \mapsto (\boldsymbol{\theta},\mathbf{J})$

$(\mathbf{x},\mathbf{v}) \mapsto (\boldsymbol{\theta},\mathbf{J})$ → *Distribution of orbits* → $F_0(\mathbf{x},\mathbf{v}) = F_0(\mathbf{J})$

$F_0(\mathbf{x},\mathbf{v}) = F_0(\mathbf{J})$ → $\int \mathrm{d}\mathbf{v}$ → $\rho_0(R,z)$

$\rho_0(R,z)$ → *Poisson Eq.* → $\Phi_0(R,z)$

Mean-field equilibrium modelling

Searching for an **equilibrium** solution

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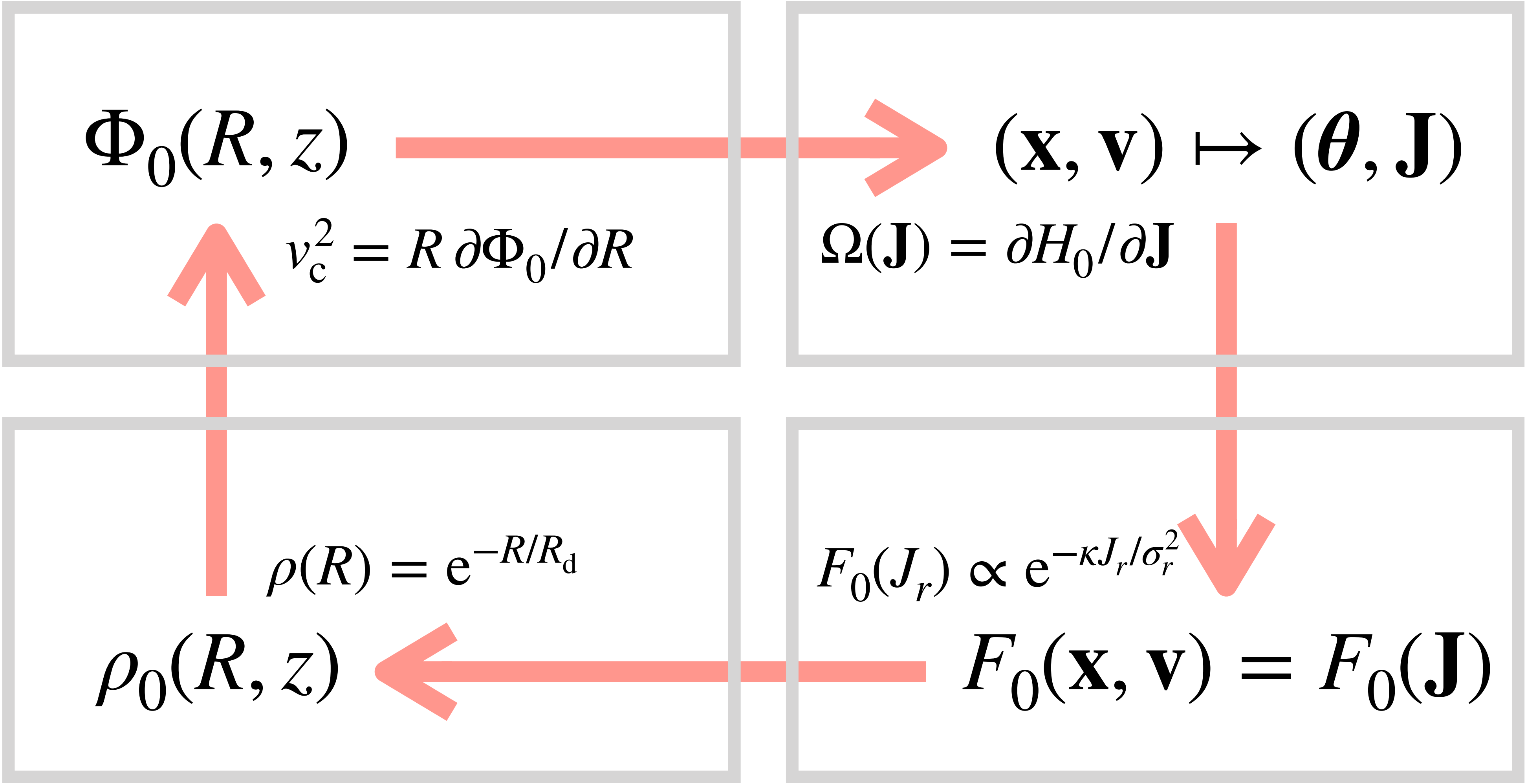


Dynamics of perturbations

Full (exact) evolution equation

$$\frac{\partial(F_0 + \delta F)}{\partial t} + \left[F_0 + \delta F, H_0 + \delta\psi^{\mathrm{ext}} + \delta\psi^{\mathrm{self}}\right] = 0$$

Mean-field equilibrium

$$\left[F_0(\mathbf{J}, t)\ ,\ H_0(\mathbf{J}, t)\right] = 0$$

First-order **perturbation theory** (**linearised Vlasov** equation)

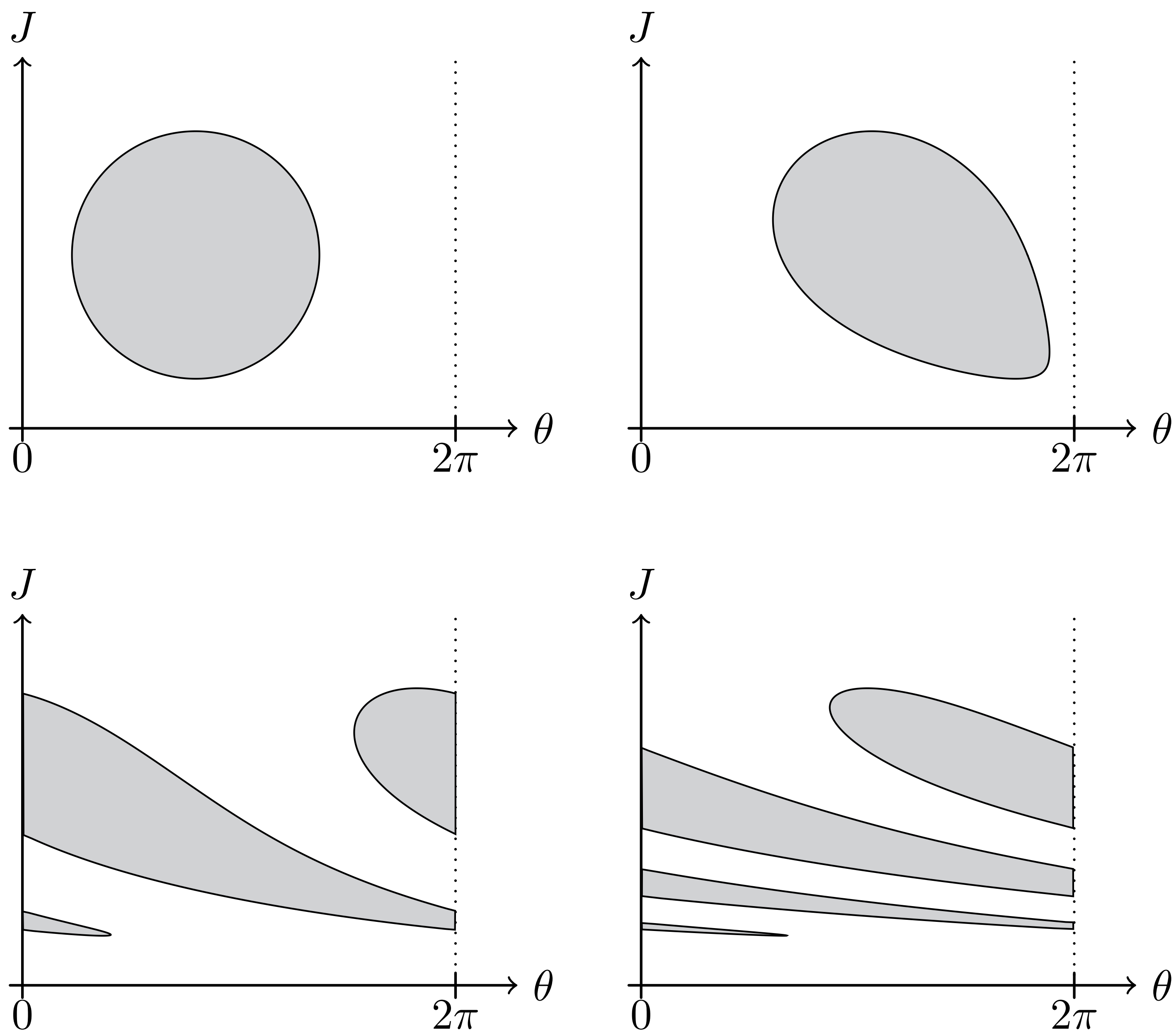
$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta\psi^{\mathrm{ext}}\right] + \left[F_0, \delta\psi^{\mathrm{self}}\right] = 0$$

Dynamics of perturbations

Phase Mixing

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] = 0$$

Shearing in angle-space

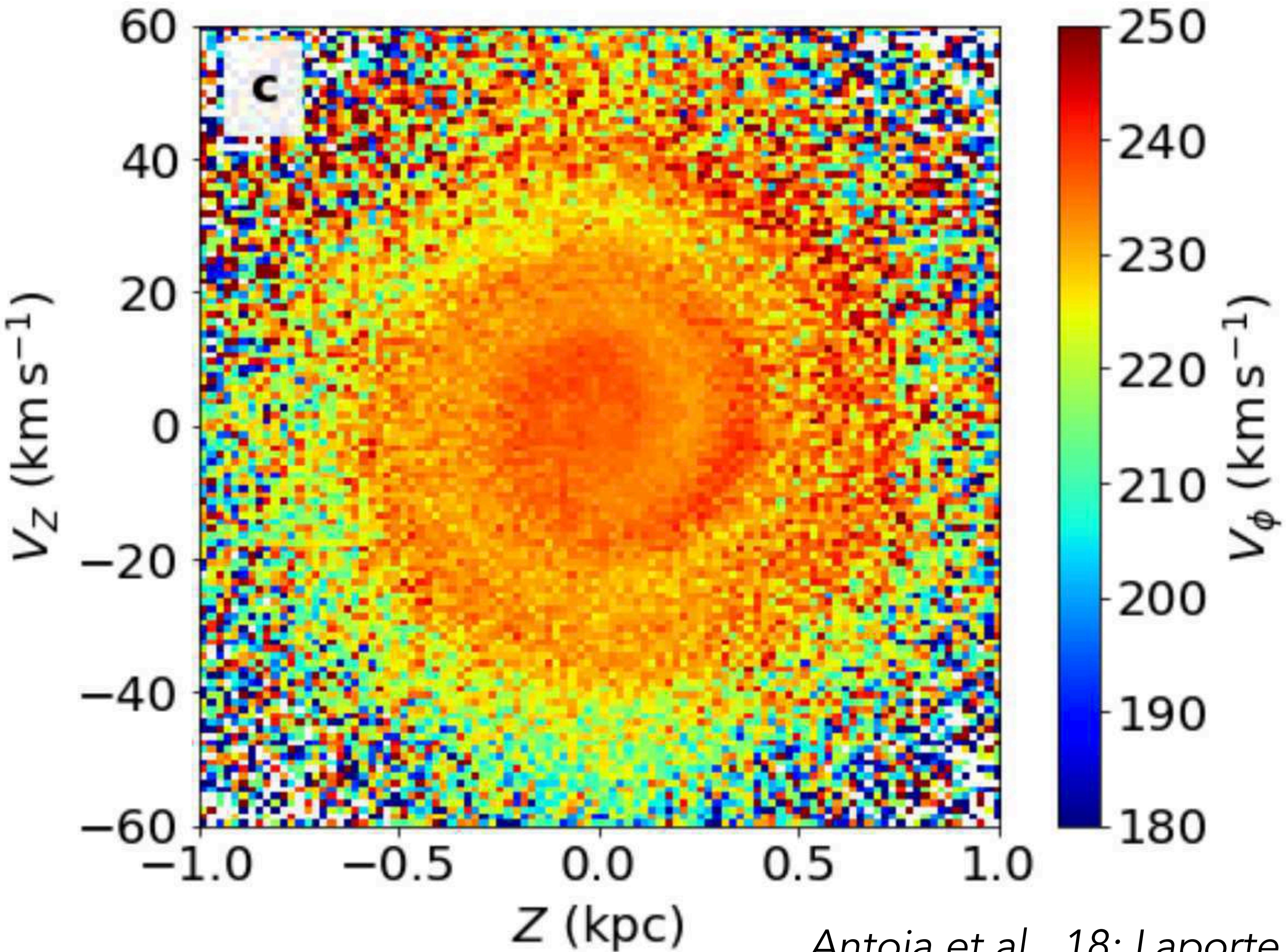


Dynamics of perturbations

Phase Mixing, i.e. *shearing in angle-space*

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] = 0$$

Phase space spiral



Antoja et al., 18; Laporte et al., 19

Dynamics of perturbations

Response to **external** perturbations

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta\psi^{\mathrm{ext}}\right] = 0$$

Possible **resonant matchings**

$$\frac{\partial \delta F}{\partial t} + \frac{\partial \delta F}{\partial \boldsymbol{\theta}} \cdot \boldsymbol{\Omega}(\mathbf{J}) + \frac{\partial F_0}{\partial \mathbf{J}} \cdot \frac{\partial \delta\left(\psi^{\mathrm{ext}} \mathrm{e}^{i m_{\mathrm{p}} \Omega_{\mathrm{p}} t}\right)}{\partial \boldsymbol{\theta}} = 0$$

Resonant matching where

$$\mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}) = m_{\mathrm{p}} \Omega_{\mathrm{p}}$$

see, e.g. Monari et al.; Fragkoudi et al.; Trick et al. 2019

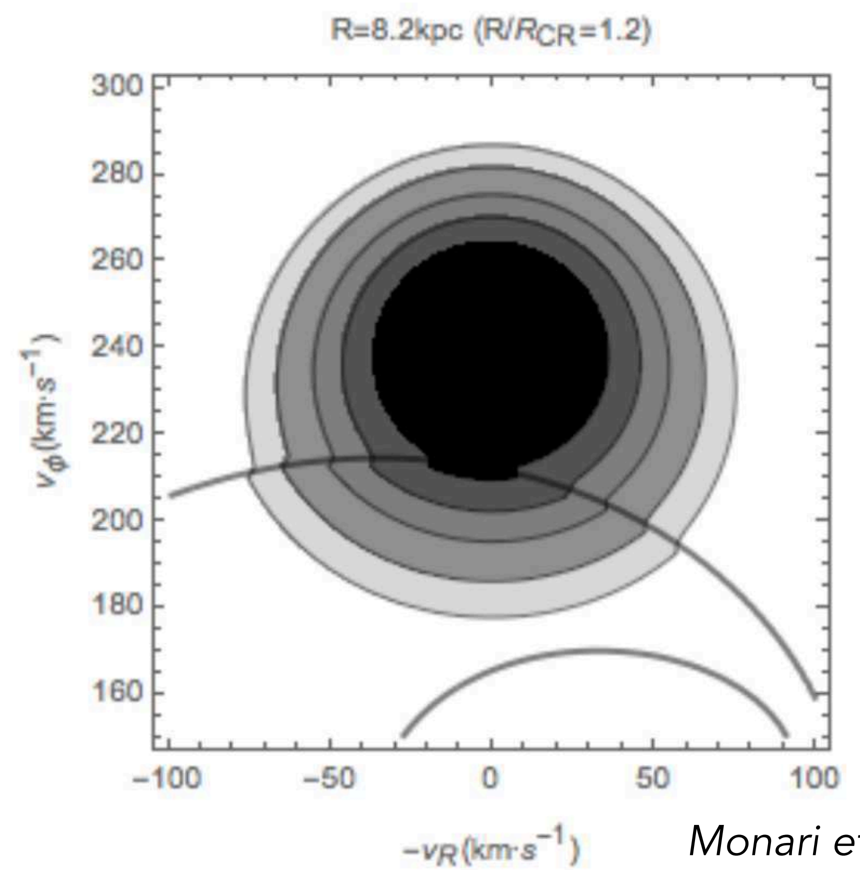
Setup identical to

+ Test particle simulations

e.g., Kamdar et al. 2019

+ Backward integration

Dehnen, 2000



Monari et al. 2017

Dynamics of dressed perturbations

Dressed response to **external** perturbations

$$\frac{\partial \delta F}{\partial t} + \left[\delta F, H_0\right] + \left[F_0, \delta\psi^{\mathrm{ext}}\right] + \left[F_0, \delta\psi^{\mathrm{self}}\right] = 0$$

Self-consistency

$$\delta\psi^{\mathrm{self}}(\mathbf{x}) = -G \int \mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{v}' \frac{\delta F(\mathbf{x}', \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|}$$

Non-Markovian **amplification**

$$\delta\psi^{\mathrm{self}}(t) = \int_{-\infty}^{t} \mathrm{d}t' \, M(t - t') \left[\delta\psi^{\mathrm{ext}}(t') + \delta\psi^{\mathrm{self}}(t')\right]$$

Response Matrix

Dynamics of dressed perturbations

Dressed response to **external** perturbations

$$\frac{\partial \delta F}{\partial t} + [\delta F, H_0] + [F_0, \delta\psi^{\mathrm{ext}}] + [F_0, \delta\psi^{\mathrm{self}}] = 0$$

Self-consistency

$$\delta\psi^{\mathrm{self}}(\mathbf{x}) = -G \int \mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{v}' \frac{\delta F(\mathbf{x}', \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|}$$

Perturbations are **dressed**

$$\left[\delta\psi^{\mathrm{self}} + \delta\psi^{\mathrm{ext}}\right](\omega) = \frac{\delta\psi^{\mathrm{ext}}(\omega)}{|\varepsilon(\omega)|}$$

Dielectric function

Linear response theory

$$\varepsilon_{pq}(\omega) = 1 - \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \, \frac{\mathbf{k}\cdot\partial F_\star/\partial \mathbf{J}}{\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})} \, \psi_{\mathbf{k}}^{(p)*}(\mathbf{J}) \, \psi_{\mathbf{k}}^{(q)}(\mathbf{J})$$

Self-gravitating amplification

Dielectric function

$$\varepsilon_{pq}(\omega)$$

Two limits

$\varepsilon_{pq}(\omega) \simeq 0$ *Cold regime*

$\varepsilon_{pq}(\omega) \simeq 1$ *Hot regime*

Some properties

$\sum_{\mathbf{k}}$ *Sum over resonances*

$\int \mathrm{d}\mathbf{J}$ *Scan over orbital space*

$\omega - \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})$ *Resonant int.*

$\Delta\psi^{(p)} = 4\pi G \rho^{(p)}$ *Gravitational int.*

Unstable modes

Unstable modes correspond to **infinite amplification**

$$\left[\delta\psi^{\mathrm{self}} + \delta\psi^{\mathrm{ext}}\right](\omega) = \frac{\delta\psi^{\mathrm{ext}}(\omega)}{|\,\varepsilon(\omega)\,|}$$

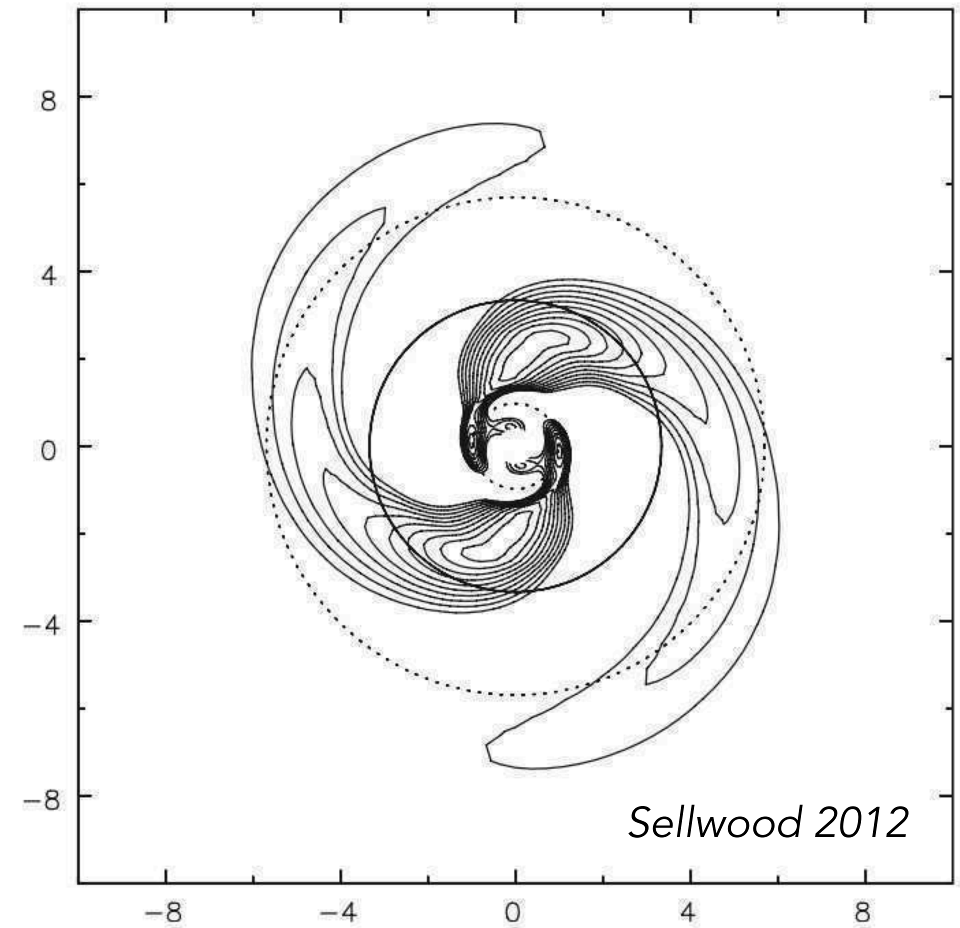
Unstable mode $\varepsilon(m_{\mathrm{p}}\Omega_{\mathrm{p}} + \mathrm{i}\eta) = 0$ $\begin{cases} \Omega_{\mathrm{p}} & \textit{Pattern speed} \\ \eta & \textit{Growth rate} \end{cases}$

Exponential growth (up to saturation)

$$\delta\psi^{\mathrm{self}}(t) \propto \mathrm{e}^{\mathrm{i}m_{\mathrm{p}}(\phi-\Omega_{\mathrm{p}}t)}\,\mathrm{e}^{\eta t}$$

Difficult to characterise

+ *Angle-action coordinates*
+ *Resonances*
+ *Basis elements*



Sellwood 2012

Dispersion relation

Self-gravity is also important for **stable systems**

$$\left[\delta\psi^{\mathrm{self}} + \delta\psi^{\mathrm{ext}}\right](\omega) = \frac{\delta\psi^{\mathrm{ext}}(\omega)}{|\,\varepsilon(\omega)\,|}$$

Difficulty: **non-locality** of Poisson equation

$$\delta\psi^{\mathrm{self}} \overset{\text{Poisson}}{\Longleftrightarrow} \delta\Sigma^{\mathrm{self}} = \int \mathrm{d}\mathbf{v}\, \delta F$$

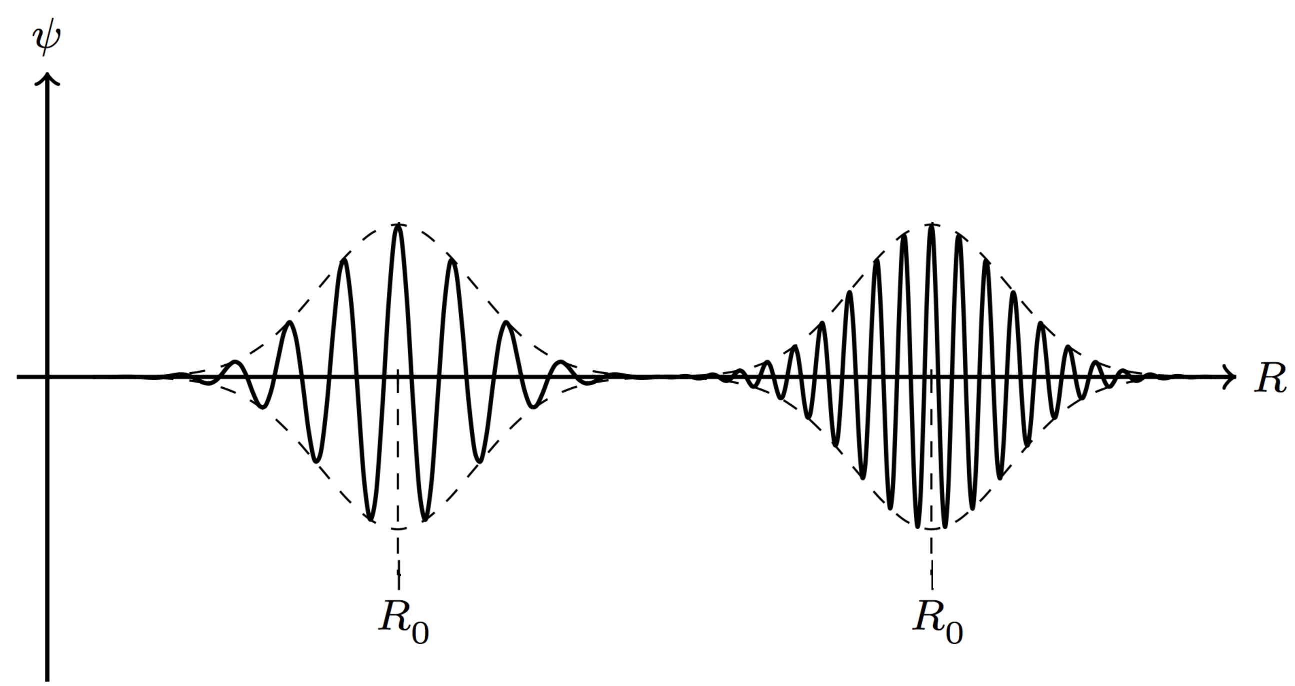
WKB limit (i.e. tightly wound waves)

$$\delta\psi^{\mathrm{ext}} \propto \mathrm{e}^{\mathrm{i}(m_{\mathrm{p}}\phi - \omega)t}\, \mathrm{e}^{\mathrm{i}k_r R}$$

$(k_r R \gg 1)$

Poisson becomes **local**

$$\delta\Sigma^{\mathrm{ext}} = -\frac{|\,k_r\,|}{2\pi G}\,\delta\psi^{\mathrm{ext}}$$



WKB dispersion relation

How much do **tightly wound waves** get amplify?

$$\delta\psi^{\mathrm{tot}} = \frac{1}{\varepsilon_{\mathrm{WKB}}}\,\delta\psi^{\mathrm{ext}}\big[R_0, m_{\mathrm{p}}, k_r, \omega\big]$$

WKB **amplification** eigenvalues

$$\varepsilon_{\mathrm{WKB}} = 1 - \frac{2\pi G\Sigma_\star\,|\,k_r\,|}{\kappa^2 - (\omega - m_{\mathrm{p}}\Omega_\phi)^2}\,\underbrace{\mathscr{F}(\omega, \sigma_r)}_{\textit{reduction factor}}$$

Disc is stable to **WKB perturbations**

$$\varepsilon_{\mathrm{WKB}}\big[R_0, m_{\mathrm{p}}, k_r, \omega, \eta > 0\big] = 0$$

has no solutions

WKB dispersion relation

How much do **tightly wound waves** get amplify?

$$\delta\psi^{\mathrm{tot}} = \frac{1}{\varepsilon_{\mathrm{WKB}}}\,\delta\psi^{\mathrm{ext}}\left[R_0, m_{\mathrm{p}}, k_r, \omega\right]$$

WKB **amplification** eigenvalues

$$\varepsilon_{\mathrm{WKB}} = 1 - \frac{2\pi G\Sigma_{\star}\,|\,k_r\,|}{\kappa^2 - (\omega - m_{\mathrm{p}}\Omega_{\phi})^2}\,\mathscr{F}(\omega, \sigma_r)$$

reduction factor

Disc is stable to axisymmetric $(m_{\mathrm{p}} = 0)$ WKB perturbations

$$Q = \frac{\sigma_r\,\kappa}{3.36\,G\,\Sigma_{\star}} > 1$$

But not all waves are tightly wound

What happens when $k_r R \to 0$?

$$\delta\psi^{\text{self}}(t) = \int_{-\infty}^{t} \mathrm{d}t' \, M(t-t') \left[\delta\psi^{\text{ext}}(t') + \delta\psi^{\text{self}}(t') \right]$$

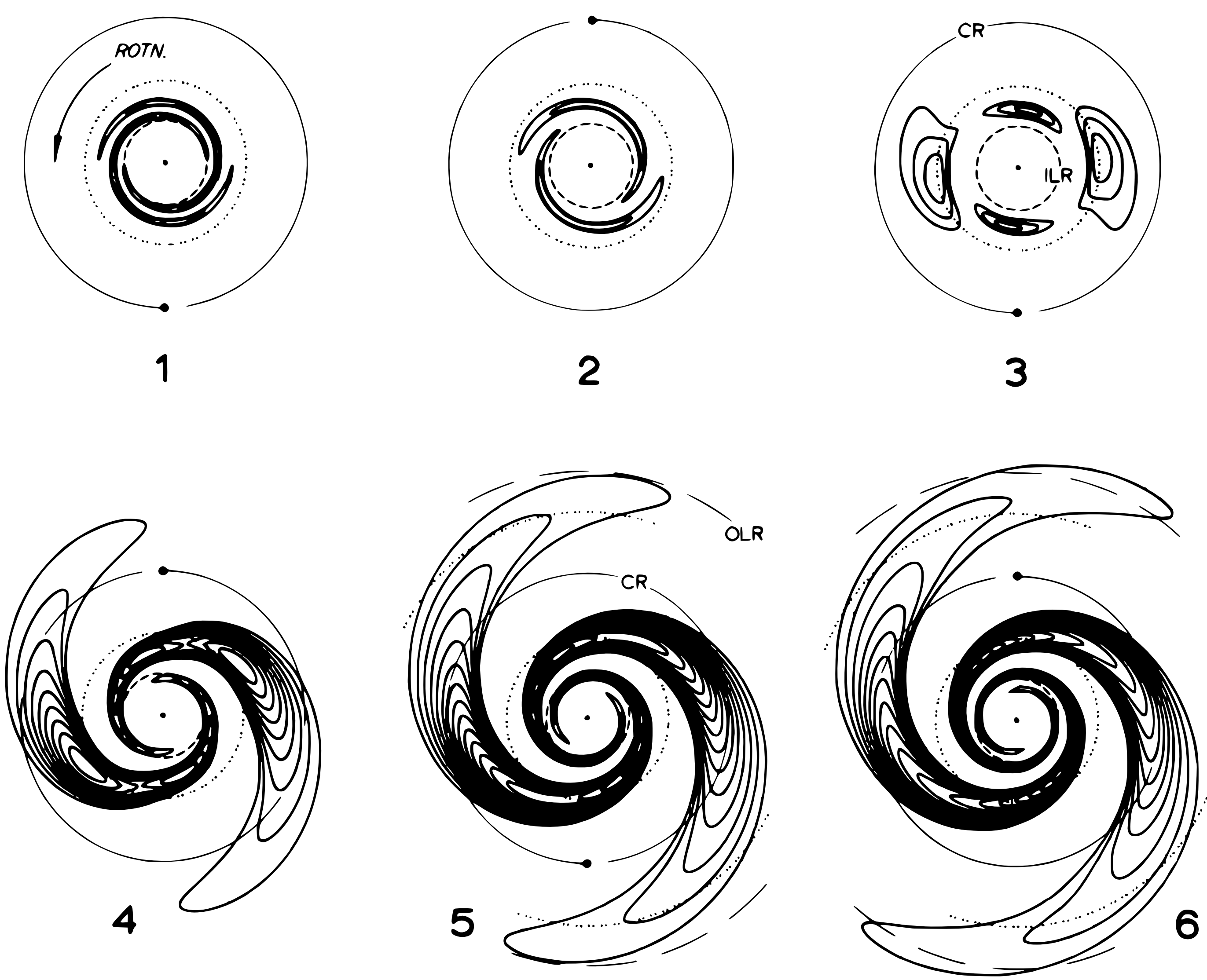
Waves naturally unwind from leading to trailing

$$\partial\Omega_\phi / \partial R < 0$$

Time-dependent **swing amplification**

Julian&Toomre, 1966
Binney, 2019

$$Q \sim 1.5 \Rightarrow \frac{1}{|\varepsilon|} \sim 30$$



Toomre, 1981

Long-term dynamics

Full (exact) evolution equation

$$\frac{\partial(F_0 + \delta F)}{\partial t} + \left[F_0 + \delta F, H_0 + \delta\psi^{\mathrm{ext}} + \delta\psi^{\mathrm{self}}\right] = 0$$

Mean-field equilibrium

$$\left[F_0(\mathbf{J}, t)\ ,\ H_0(\mathbf{J}, t)\right] = 0$$

Second-order **perturbation theory**

$$\frac{\partial F_0(\mathbf{J}, t)}{\partial t} + \left\langle \left[\delta F, \delta\psi^{\mathrm{ext}} + \delta\psi^{\mathrm{self}}\right] \right\rangle = 0$$

Dressed long-term dynamics

Secular evolution equation

$$\frac{\partial F_0(\mathbf{J},t)}{\partial t} + \left\langle \left[\delta F, \delta\psi^{\mathrm{tot}}\right] \right\rangle = 0$$

Dressing comes twice

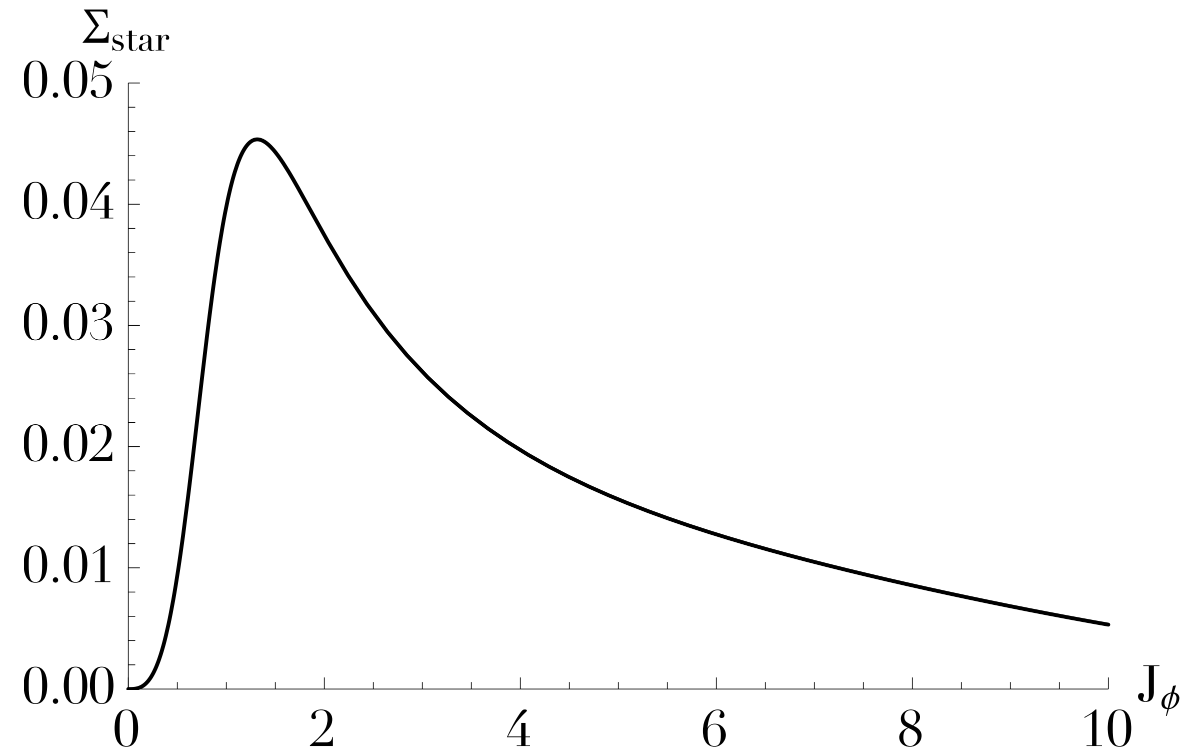
$$\delta\psi^{\mathrm{tot}} \propto \frac{\delta\psi^{\mathrm{ext}}}{|\,\varepsilon(\omega)\,|} \longrightarrow \frac{\partial F_0(\mathbf{J},t)}{\partial t} \propto \frac{|\,\delta\psi^{\mathrm{ext}}\,|^2}{|\,\varepsilon\,[\Omega(\mathbf{J})]\,|^2}$$

Collective effects can **drastically accelerate** orbital heating, e.g. through **swing amplification**

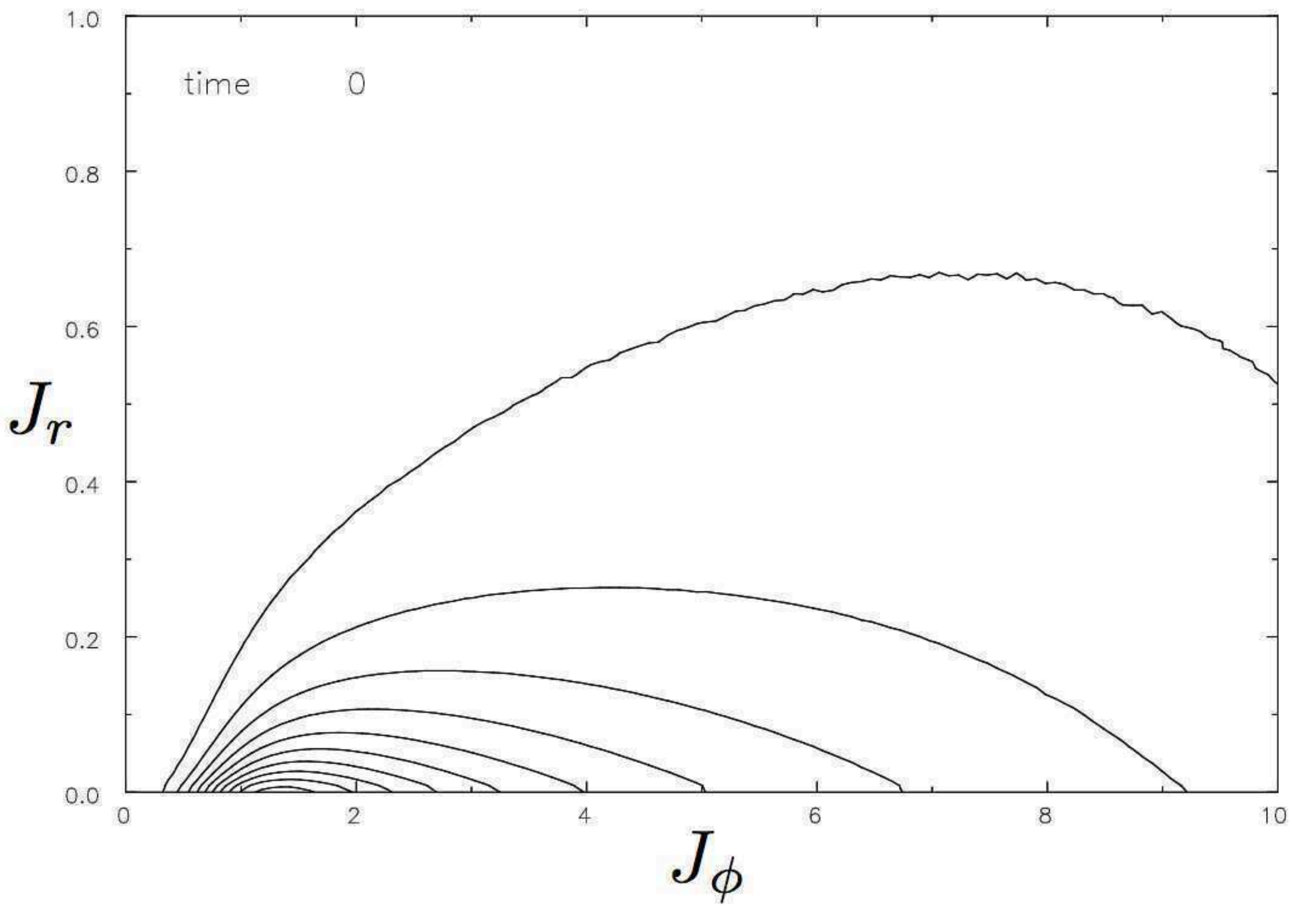
An example of secular evolution

Sellwood 2012's numerical experiment

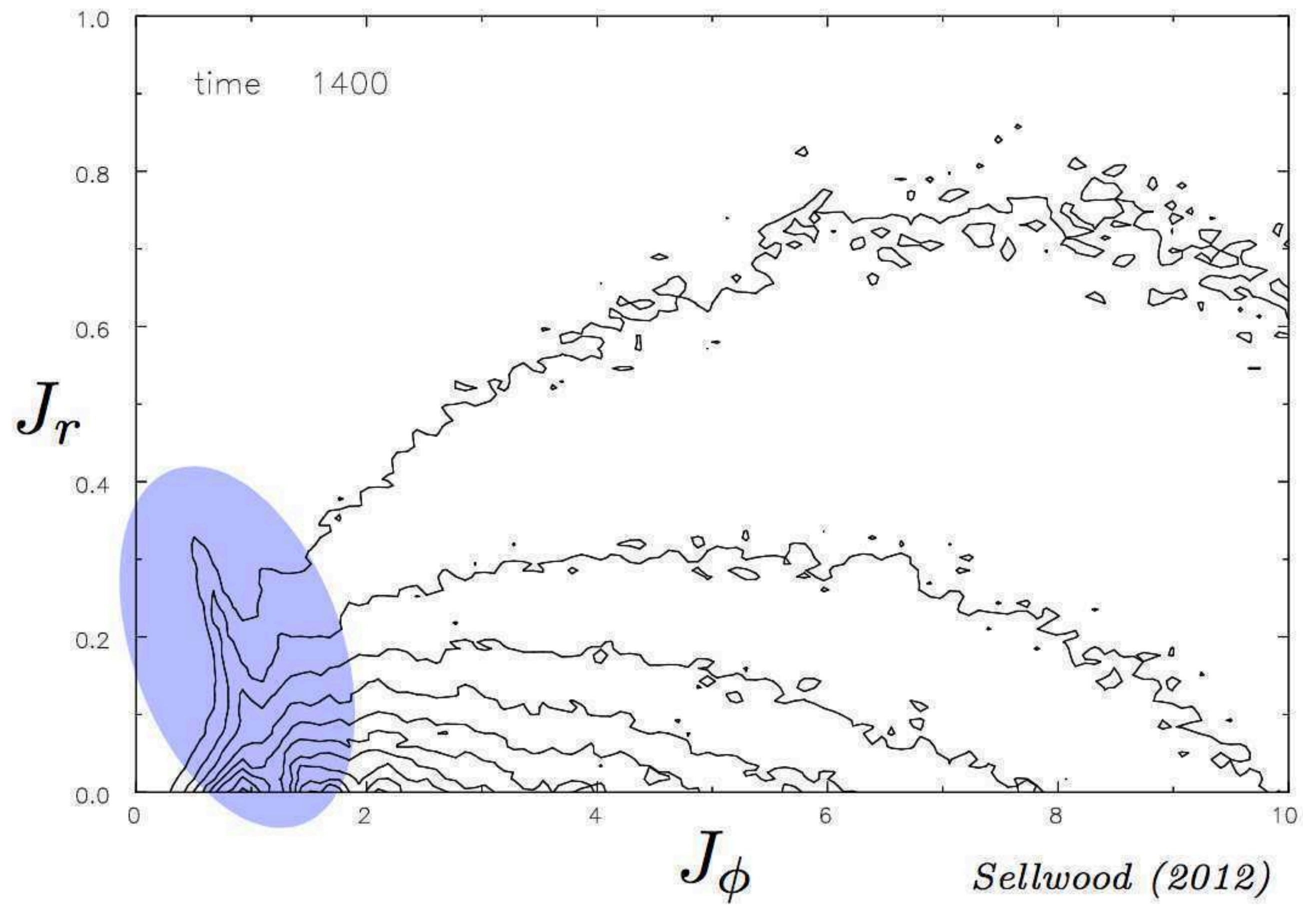
+ Stationary **stable** Mestel disc
+ Sampled with **500M particles**
+ Unavoidable **transient waves**



Initial **stable/stationary** DF



Evolved DF



Sellwood (2012)

In configuration space

Some remarks

+ Nothing spectacular happens

Linearly stable

+ Cannot track **disc's heating**

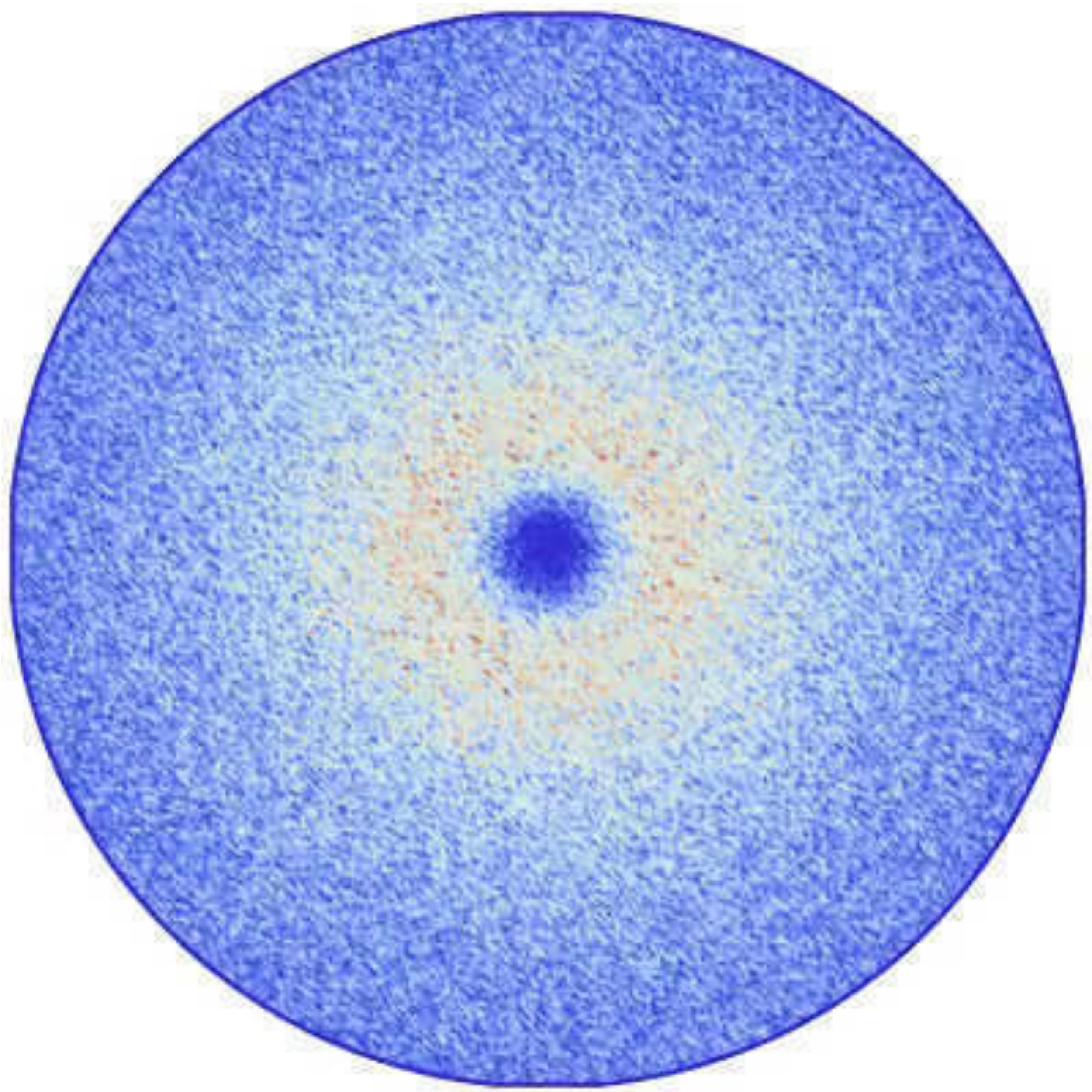
Inhomogeneous

+ **Large** transients

Self-gravitating

+ Fluctuations are **absorbed**

Resonant



In action space

Some remarks

+ **Heating** in orbital space

Inhomogeneous

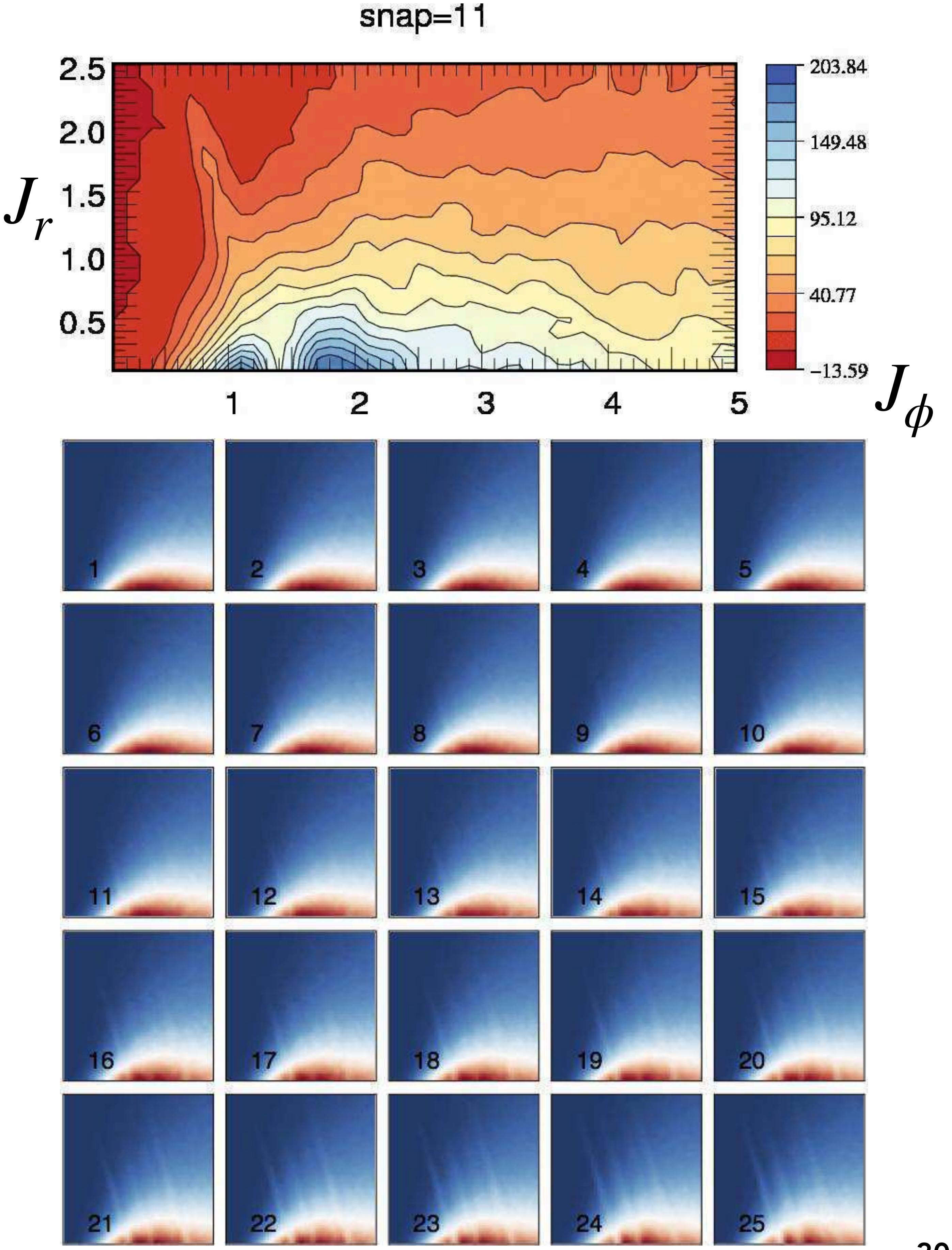
+ **Fast** heating

$$T_{\mathrm{ridge}} \simeq |\varepsilon|^2 N\, T_{\mathrm{dyn}}$$

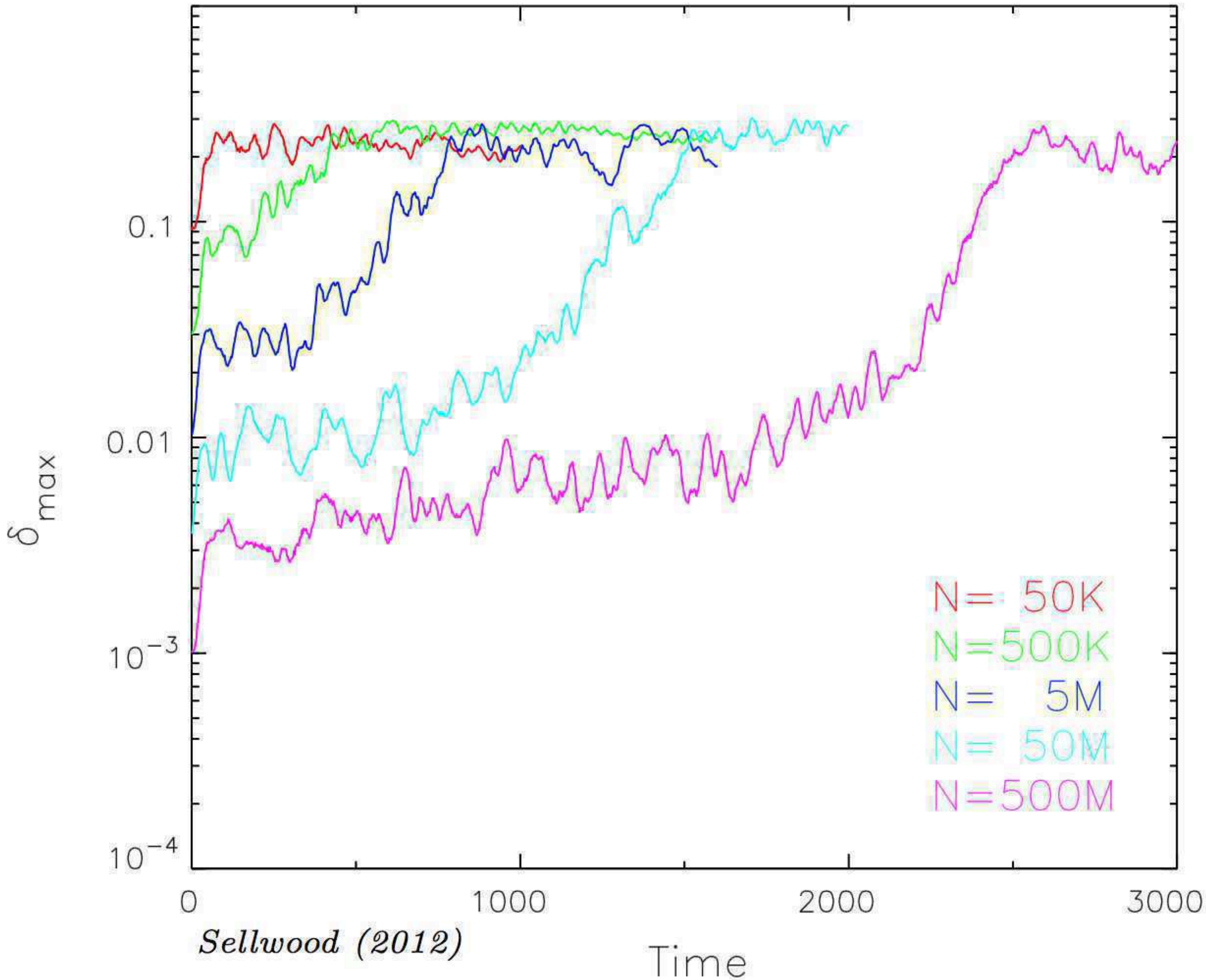
Self-gravitating

+ **Localised heating**

Resonant



A dynamics sourced by finite-N effects



The larger the number of particles, the slower the effect

Needed ingredients

Needed ingredients for that dynamics

+ Disc is frozen on **dynamical times**

Linearly stable

+ Disc is **isolated**

No external perturbations

+ Disc has **internal** fluctuations

Internal Poisson noise

+ Fluctuations are **large**

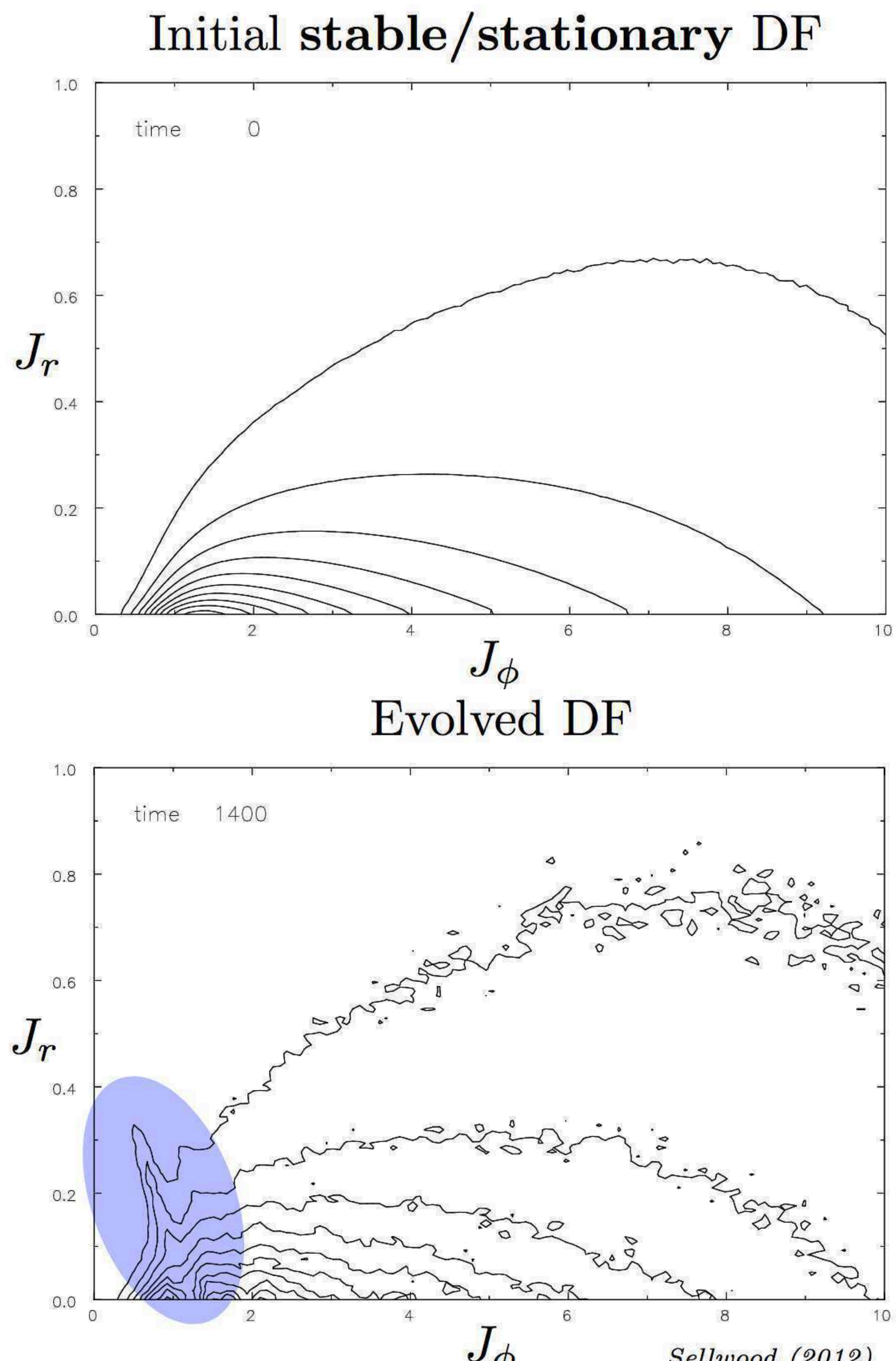
Self-gravitating

+ Heating happens in **orbital space**

Inhomogeneous

+ Heating is **localised**

Resonant



Sellwood (2012)

The Balescu-Lenard equation

The master equation for **self-induced orbital redistribution**

$$\frac{\partial F_\star(\mathbf{J}_1,t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[\sum_{\mathbf{k}_1} \mathbf{k}_1 \left\{ D^{\mathrm{fric}}_{\mathbf{k}_1}(\mathbf{J}_1)\, F_\star(\mathbf{J}_1,t) + D^{\mathrm{diff}}_{\mathbf{k}_1}(\mathbf{J}_1)\, \mathbf{k}_1 \cdot \frac{\partial F_\star(\mathbf{J}_1,t)}{\partial \mathbf{J}_1} \right\} \right]$$

Anisotropic diffusion coefficients

$$D^{\mathrm{diff}}_{\mathbf{k}_1}(\mathbf{J}_1) = \frac{1}{N_\star} \sum_{\mathbf{k}_2} \int \mathrm{d}\mathbf{J}_2\, \delta_{\mathrm{D}}(\mathbf{k}_1\cdot\boldsymbol{\Omega}(\mathbf{J}_1) - \mathbf{k}_2\cdot\boldsymbol{\Omega}(\mathbf{J}_2))\, \frac{F_\star(\mathbf{J}_2,t)}{|\varepsilon_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{J}_1,\mathbf{J}_2)|^2}$$

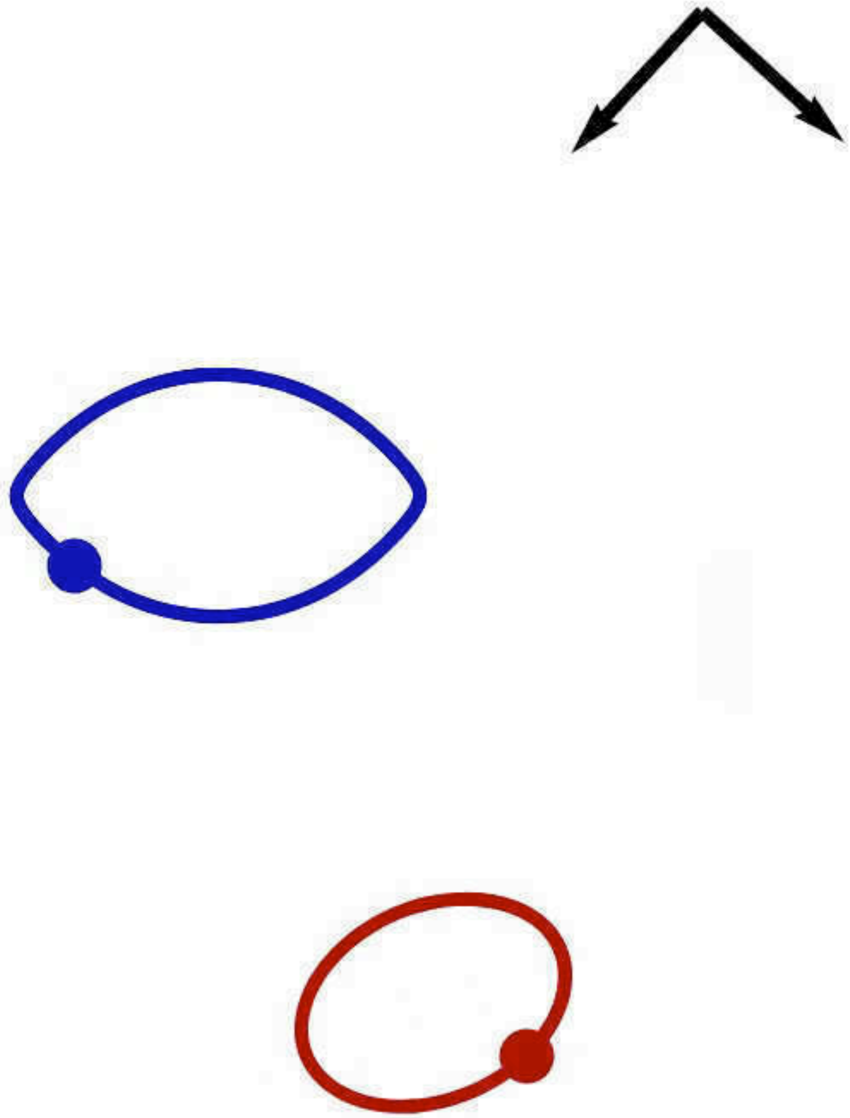
Some properties

- $F_\star(\mathbf{J},t)$ Orbital distortion
- $D^{\mathrm{fric}}_{\mathbf{k}}(\mathbf{J})$ Dynamical friction (~ cooling)
- $D^{\mathrm{diff}}_{\mathbf{k}}(\mathbf{J})$ Diffusion (~ heating)
- $1/N_\star$ Finite-N effects

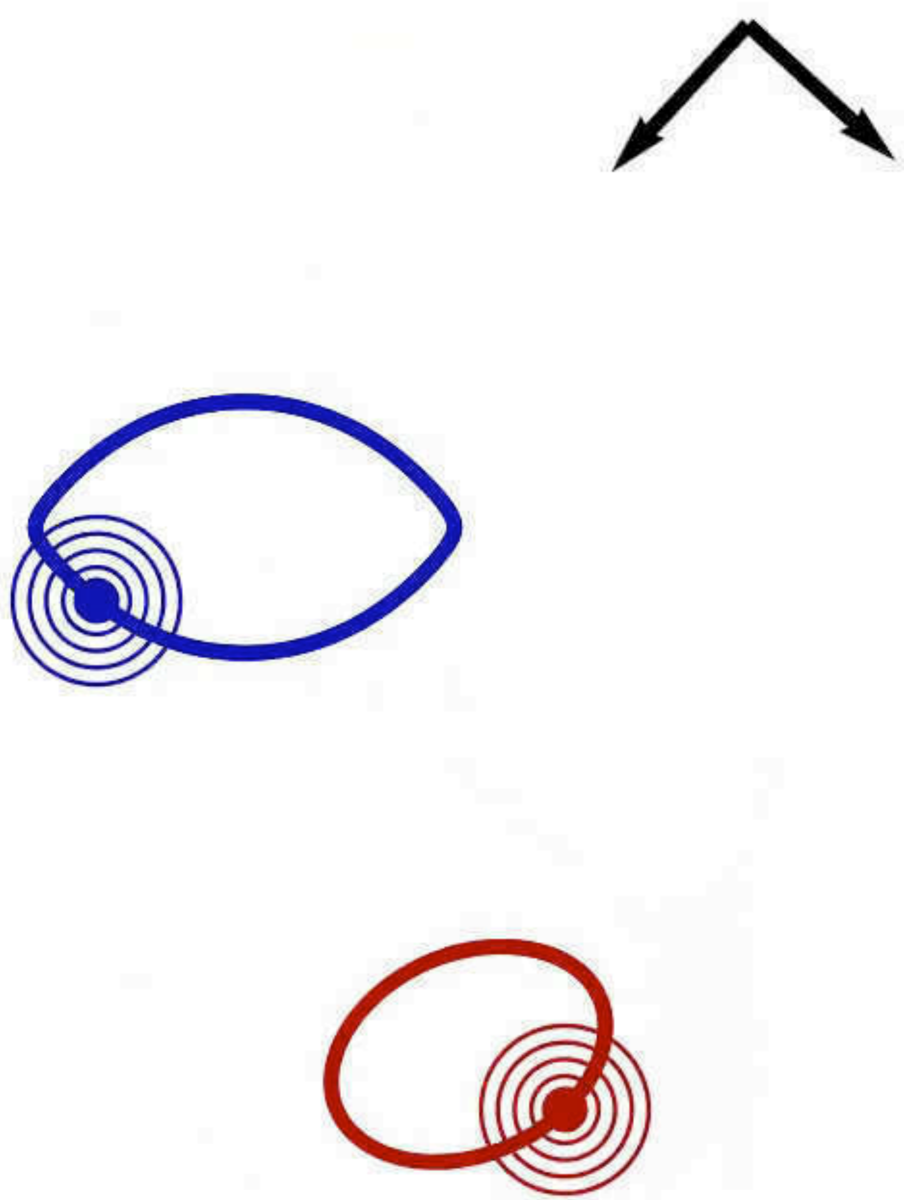
- $(\mathbf{k}_1,\mathbf{k}_2)$ Discrete resonances
- $\int \mathrm{d}\mathbf{J}_2$ Scan of orbital space
- $\delta_{\mathrm{D}}(\mathbf{k}_1\cdot\boldsymbol{\Omega}_1 - \mathbf{k}_2\cdot\boldsymbol{\Omega}_2)$ Resonance cond.
- $|\varepsilon_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{J}_1,\mathbf{J}_2)|^2$ Dressed coupling

(Resonant) encounters

$$\delta_{\mathrm{D}}\big(\mathbf{k}_1 \cdot \mathbf{\Omega}(\mathbf{J}_1) - \mathbf{k}_2 \cdot \mathbf{\Omega}(\mathbf{J}_2)\big)$$



Resonant encounters are **dressed**



Stars replaced by their **wake**

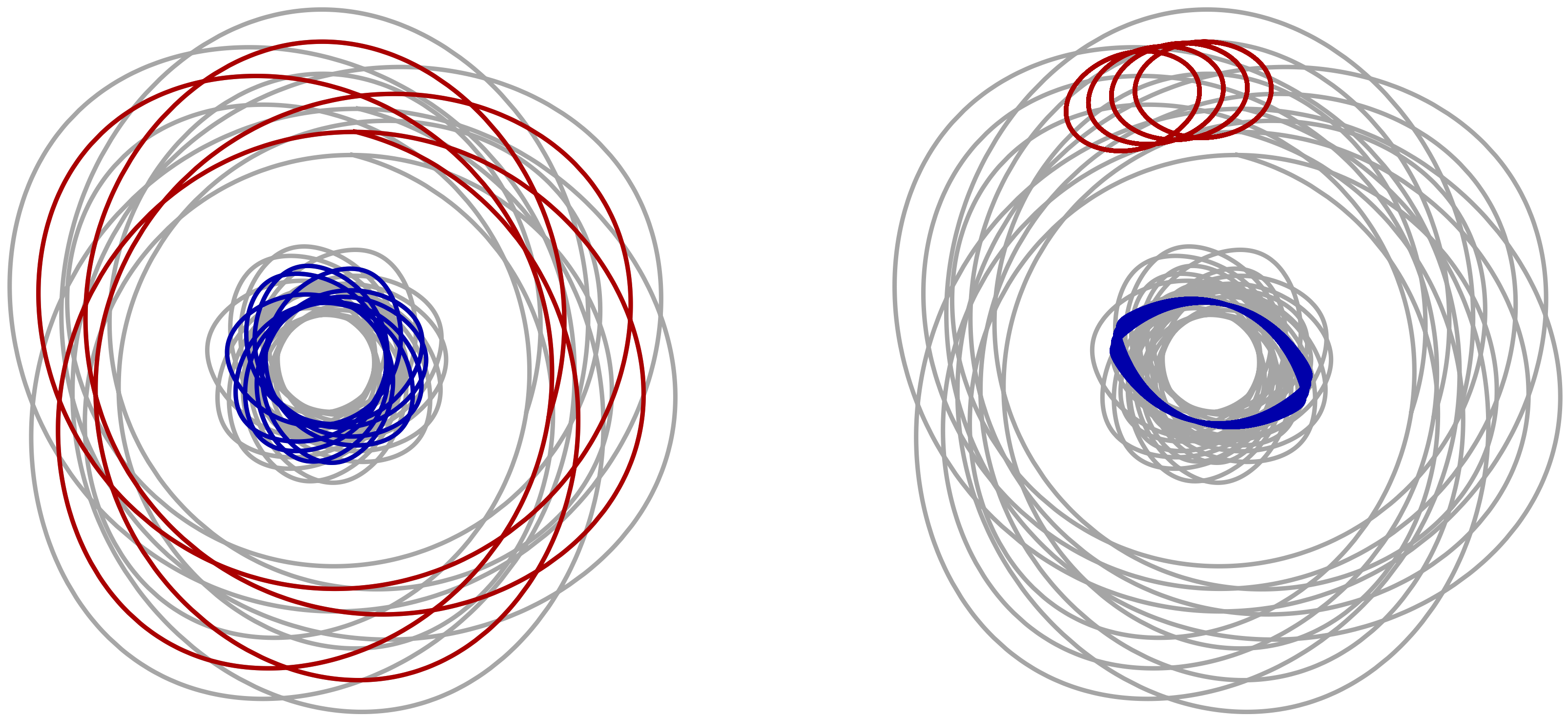
$$m_\star \to m_{\rm eff} = \frac{m_\star}{|\,\varepsilon(\Omega_\star)\,|}$$

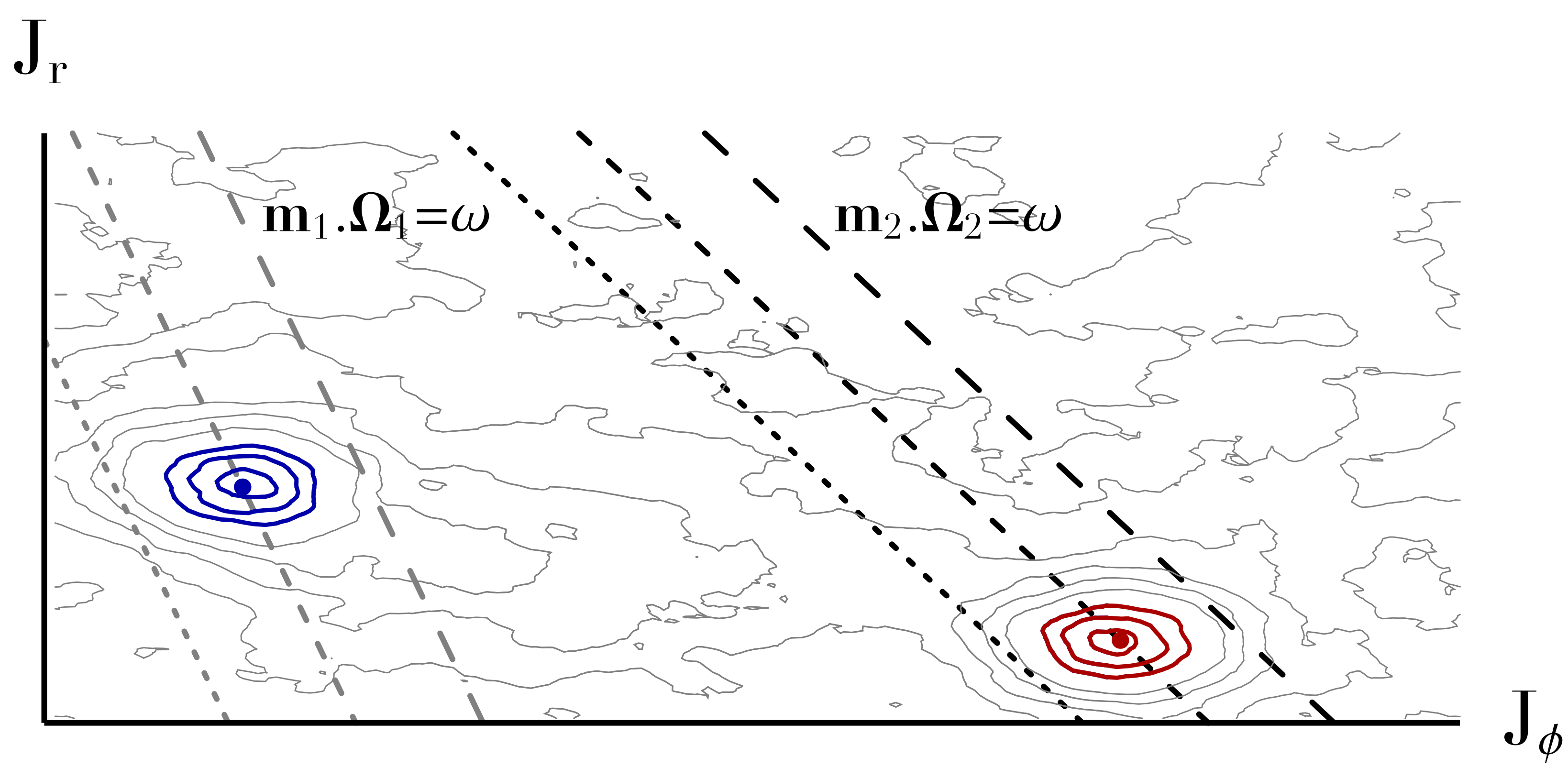
Interactions between **wakes**

$$D^{\rm diff} \propto \frac{m_\star}{|\,\varepsilon(\Omega_\star)\,|^2}$$

(Resonant) encounters

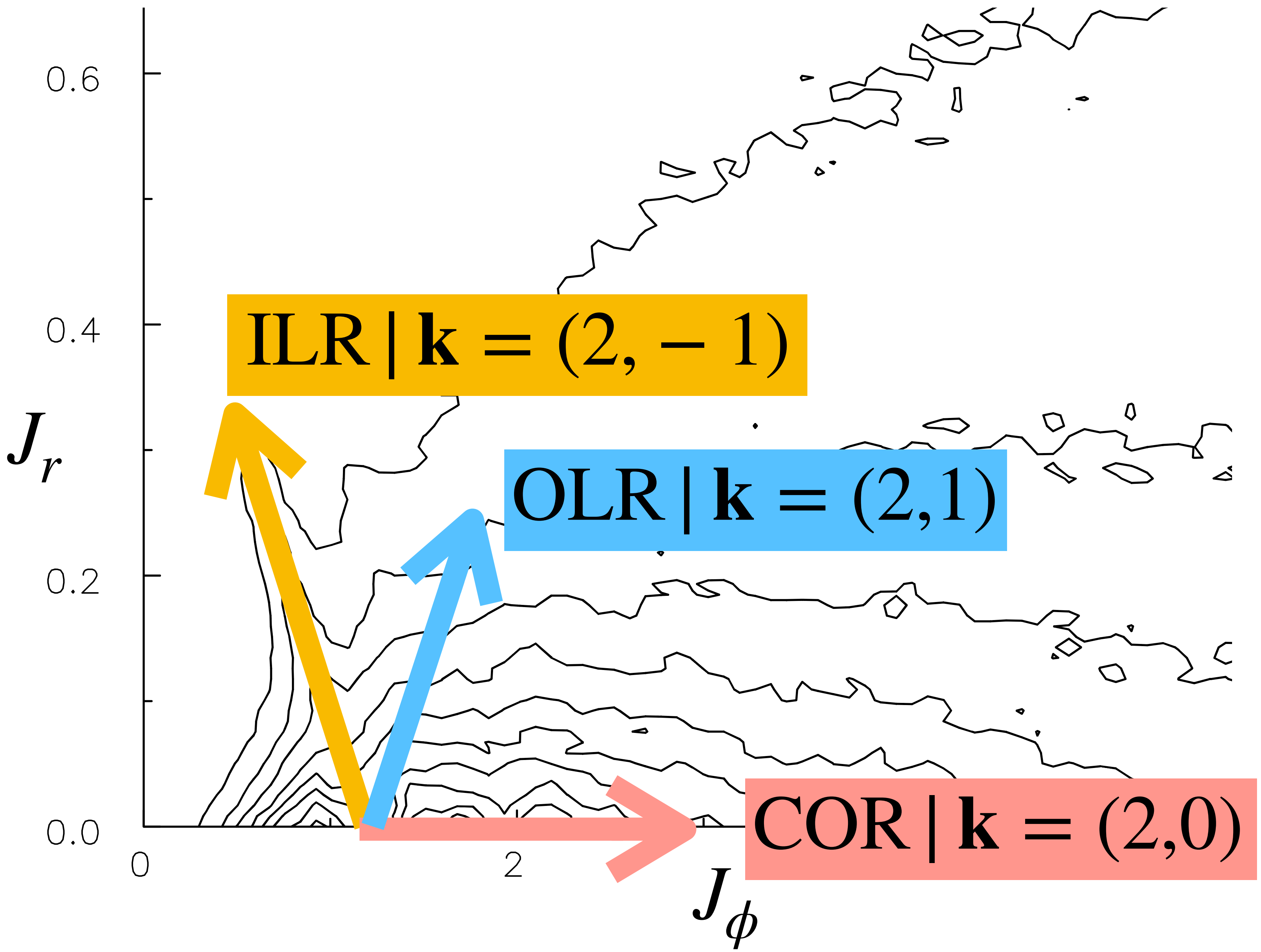
$$\delta_D\big(\mathbf{k}_1 \cdot \mathbf{\Omega}(\mathbf{J}_1) - \mathbf{k}_2 \cdot \mathbf{\Omega}(\mathbf{J}_2)\big)$$





Diffusion is anisotropic

$$\frac{\partial F_\star(\mathbf{J}_1, t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[\sum_{\mathbf{k}} \mathbf{k}\, \mathscr{F}_{\mathbf{k}}(\mathbf{J}, t) \right]$$



The Balescu-Lenard equation

The master equation for **self-induced orbital redistribution**

$$\frac{\partial F_\star(\mathbf{J}_1,t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}_1} \cdot \left[\sum_{\mathbf{k}_1} \mathbf{k}_1 \left\{ D^{\mathrm{fric}}_{\mathbf{k}_1}(\mathbf{J}_1)\, F_\star(\mathbf{J}_1,t) + D^{\mathrm{diff}}_{\mathbf{k}_1}(\mathbf{J}_1)\, \mathbf{k}_1 \cdot \frac{\partial F_\star(\mathbf{J}_1,t)}{\partial \mathbf{J}_1} \right\} \right]$$

Anisotropic diffusion coefficients

$$D^{\mathrm{diff}}_{\mathbf{k}_1}(\mathbf{J}_1) = \frac{1}{N_\star} \sum_{\mathbf{k}_2} \int \mathrm{d}\mathbf{J}_2\, \delta_{\mathrm{D}}(\mathbf{k}_1 \cdot \boldsymbol{\Omega}(\mathbf{J}_1) - \mathbf{k}_2 \cdot \boldsymbol{\Omega}(\mathbf{J}_2))\, \frac{F_\star(\mathbf{J}_2,t)}{|\,\varepsilon_{\mathbf{k}_1,\mathbf{k}_2}(\mathbf{J}_1,\mathbf{J}_2)\,|^2}$$

Difficulties

+ **Inhomogeneous** system

$$(\mathbf{x},\mathbf{v}) \mapsto (\boldsymbol{\theta},\mathbf{J})$$

+ **Long-range** interactions

$$\Delta\psi = 4\pi G\rho$$

+ **Self-gravitating**

$$|\,\varepsilon(\omega)\,|$$

+ **Resonant**

$$\delta_{\mathrm{D}}(\mathbf{k}_1 \cdot \boldsymbol{\Omega}(\mathbf{J}_1) - \mathbf{k}_2 \cdot \boldsymbol{\Omega}(\mathbf{J}_2))$$

The Balescu-Lenard equation

The master equation for **self-induced orbital redistribution**

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Difficulties

+ **Inhomogeneous** system

$$(\mathbf{x},\mathbf{v}) \mapsto (\boldsymbol{\theta},\mathbf{J})$$

Discs are **explicitly integrable**

+ **Long-range** interactions

$$\Delta\psi = 4\pi G\rho$$

Global basis elements

+ **Self-gravitating**

$$|\,\varepsilon(\omega)\,|$$

Linear response theory

+ **Resonant**

$$\delta_{\mathrm{D}}(\mathbf{k}_1 \cdot \mathbf{\Omega}(\mathbf{J}_1) - \mathbf{k}_2 \cdot \mathbf{\Omega}(\mathbf{J}_2))$$

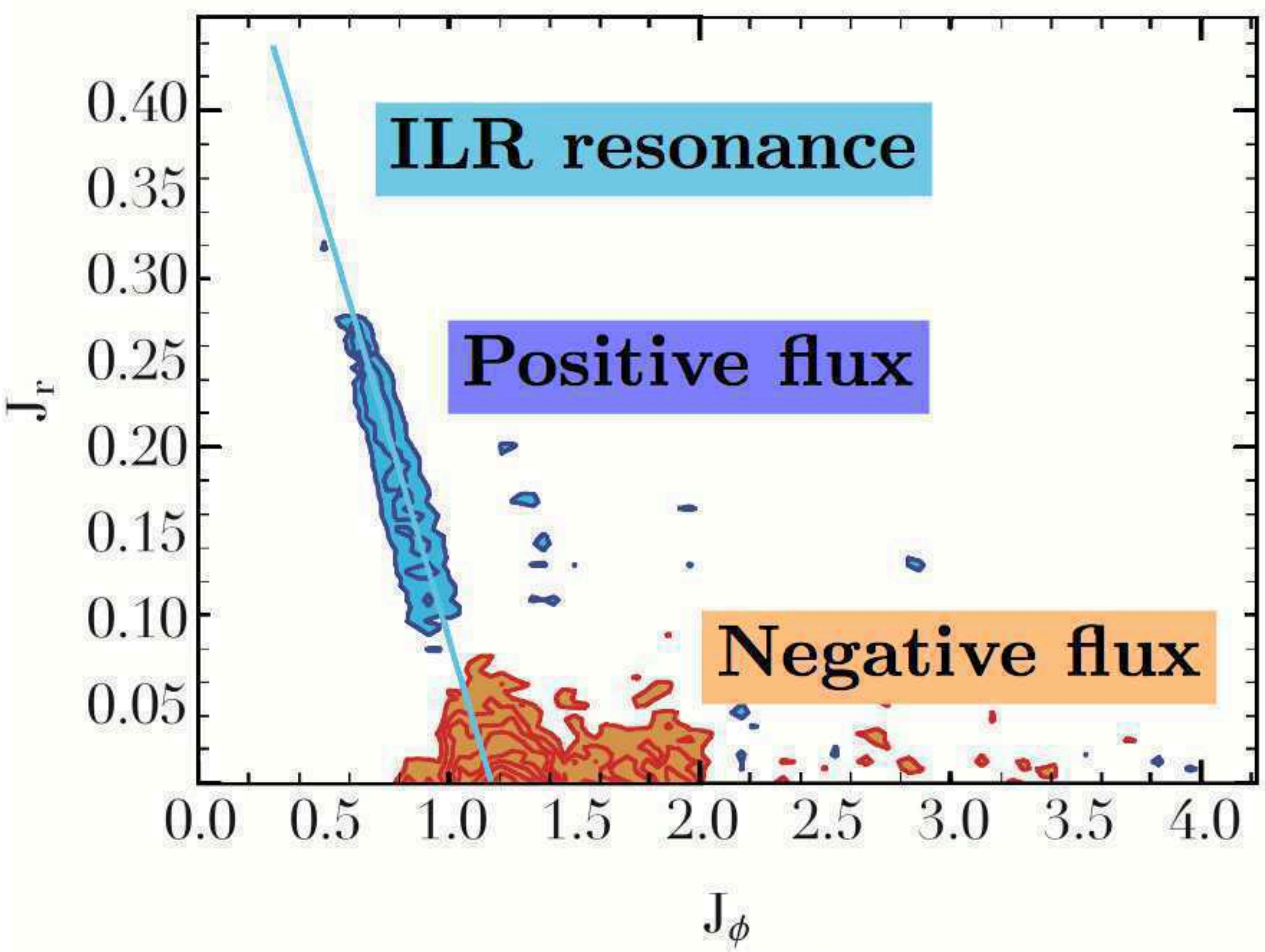
Integration along **resonant lines**

Prediction for the diffusion

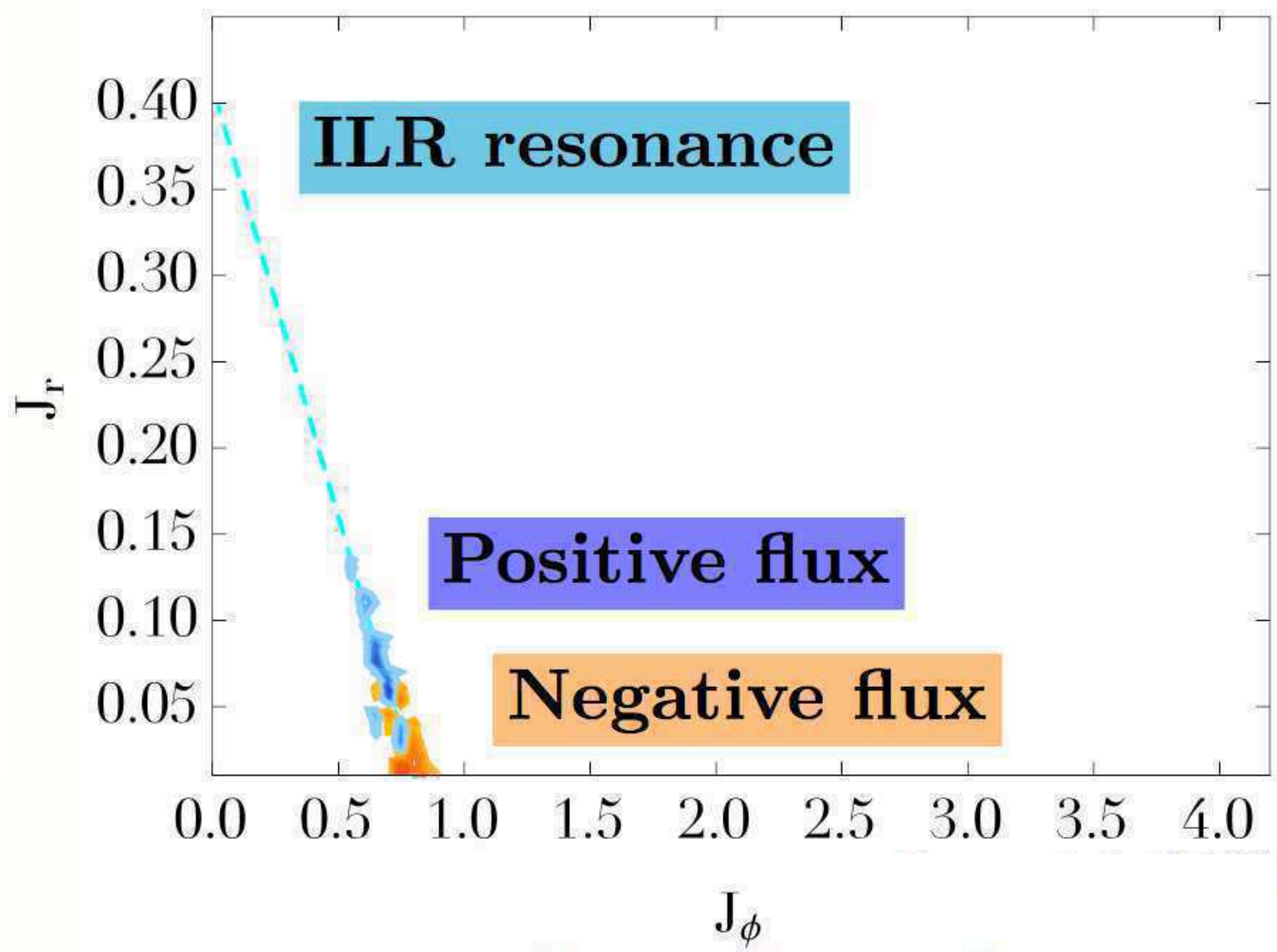
- **Diffusion flux** in action space

$$\frac{\partial F_0(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\mathscr{F}(\mathbf{J},t) \right]$$

- Predicted contours for $\mathscr{F}(\mathbf{J},t)$



Sellwood (2012)



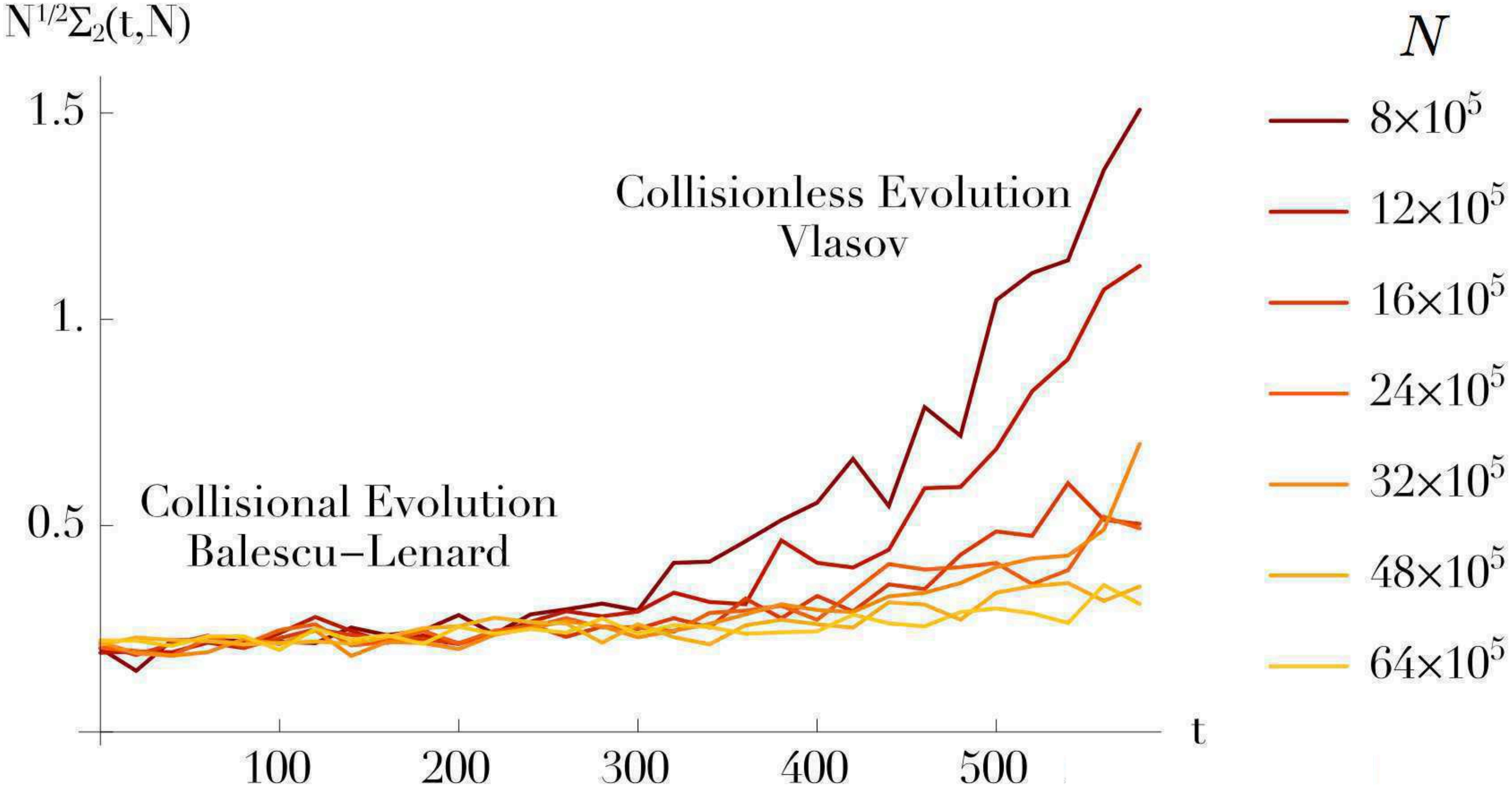
Balescu-Lenard

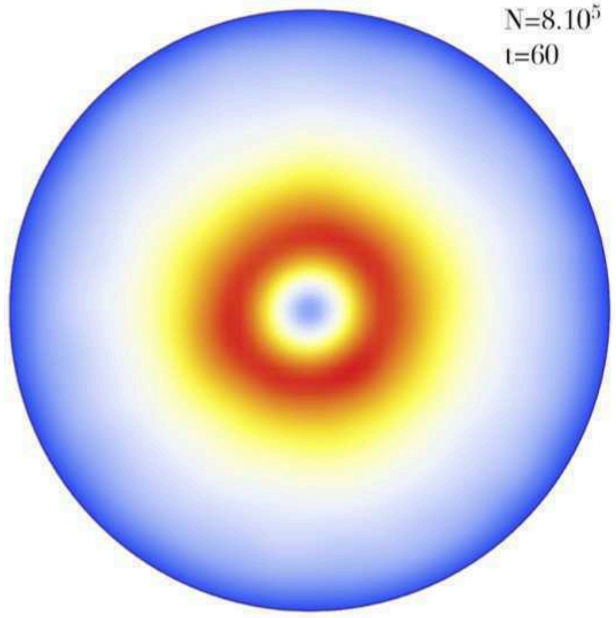
What happens after the ridge formation?

When the ridge gets large enough

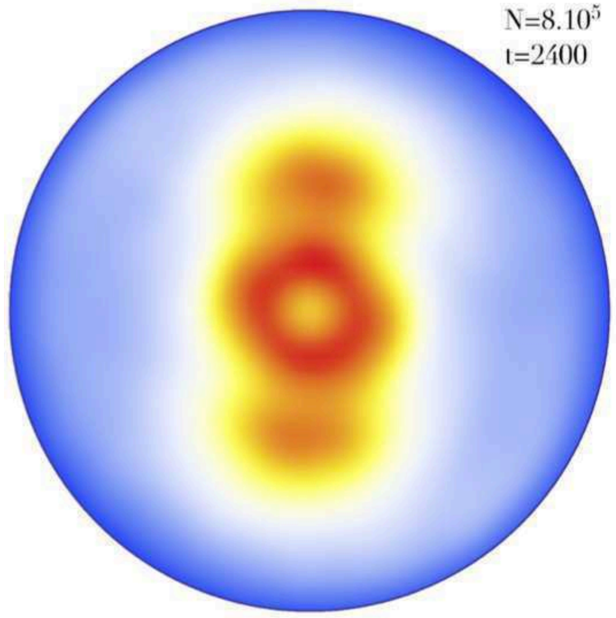
$$\frac{1}{|\varepsilon(\omega)|} = +\infty$$

Linear instability **Balescu-Lenard ⟶ Vlasov**





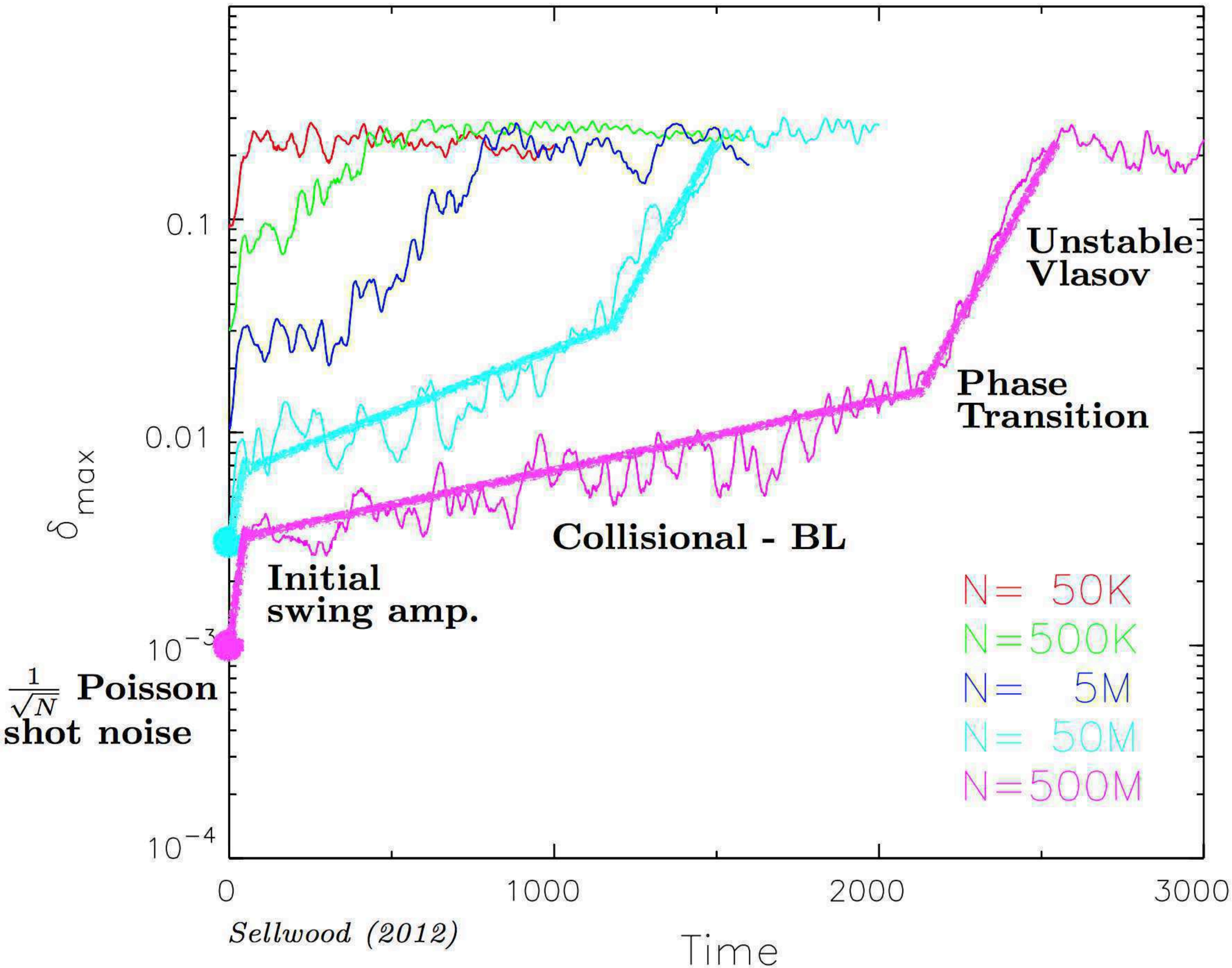
Initial times



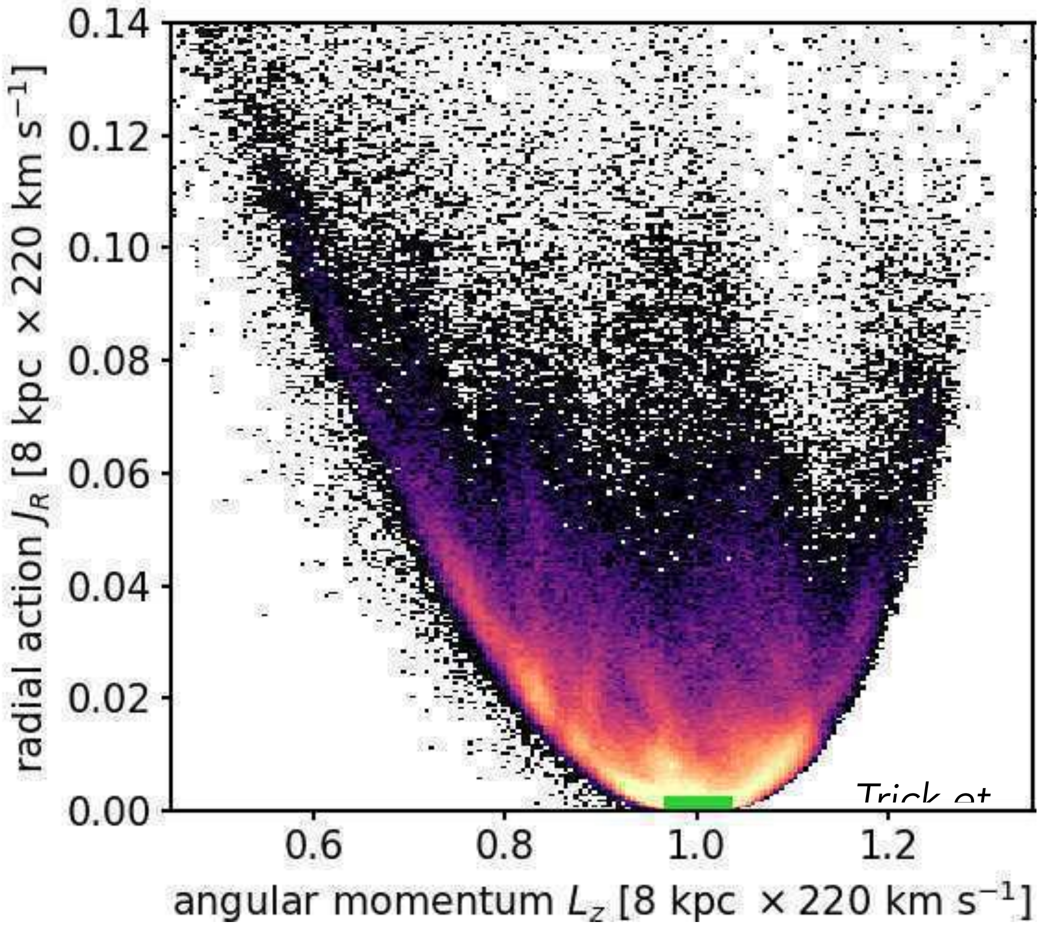
Late times

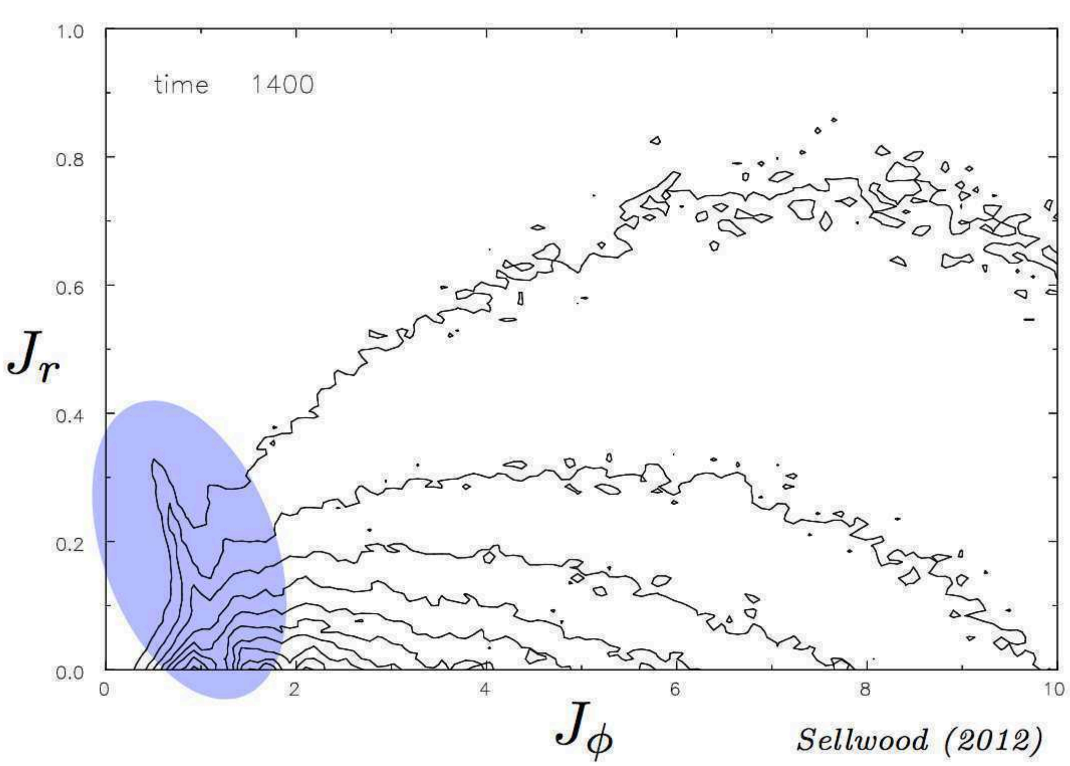
2-body **resonant** relaxation created **small-scale structures** in the DF

The fate of secular evolution



Conclusion





What is the origin of Gaia's **orbital substructures**?

+ **Dynamical** vs. **Secular**?
  e.g. transients vs. irreversible heating

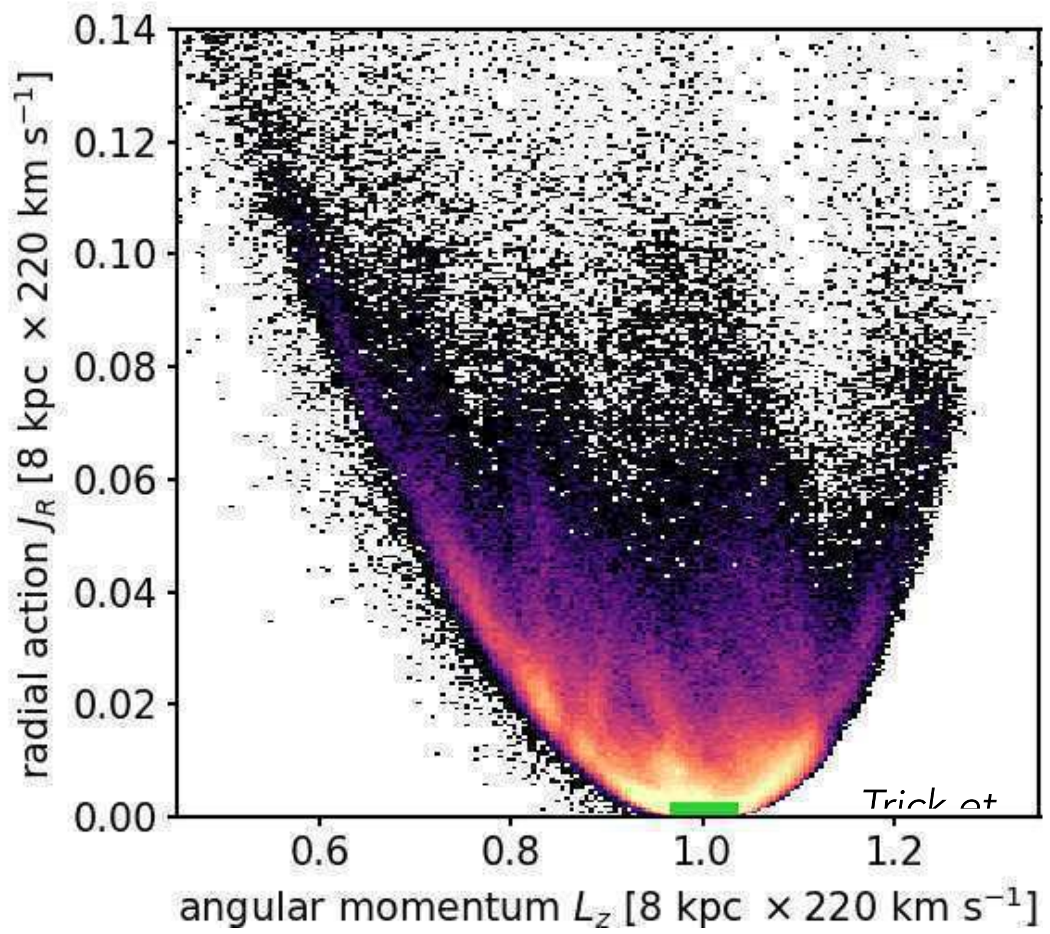
+ **External** vs. **Internal**?
  e.g. satellites vs. spiral arms

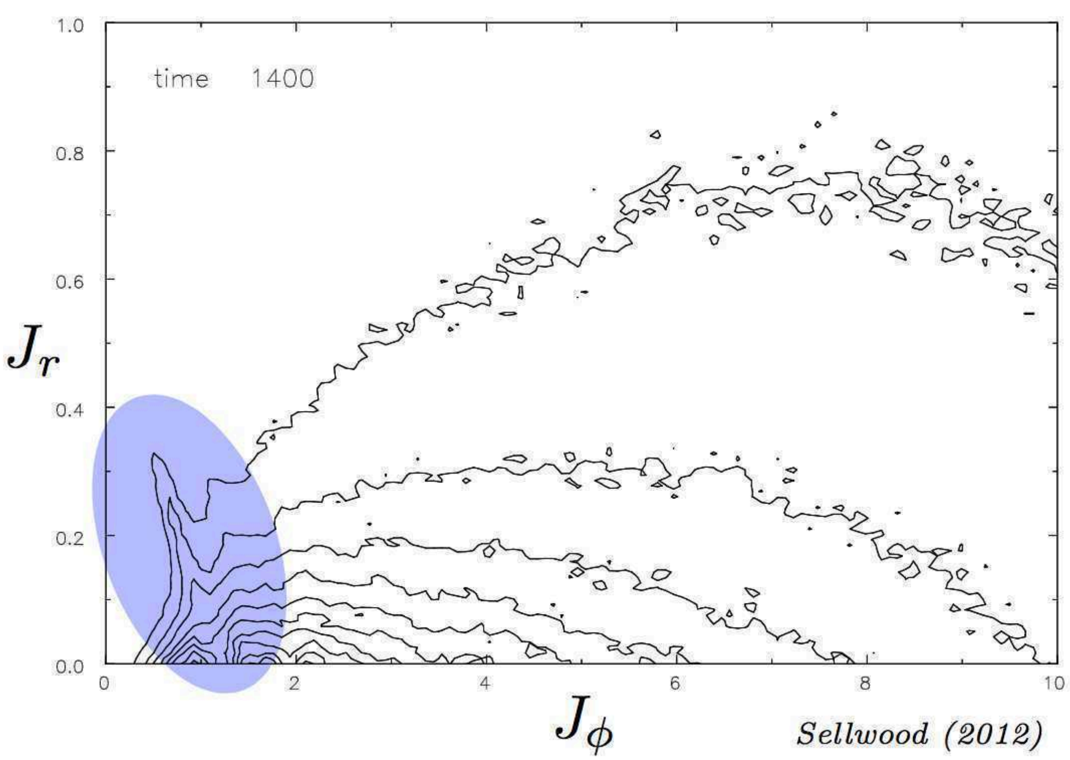
+ **Resonant** vs. **Stochastic**?
  e.g. bar trapping vs. GMCs scatterings

Three fundamental properties

Inhomogeneous	$(\mathbf{x}, \mathbf{v}) \mapsto (\boldsymbol{\theta}, \mathbf{J})$
Self-gravitating	$\lvert \varepsilon(\omega) \rvert$
Resonant	$\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) = m_{\mathrm{p}}\Omega_{\mathrm{p}}$

Conclusion





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  e.g. bar trapping vs. GMCs scatterings

Formalism not limited to galactic discs

+ **Globular clusters**

  e.g. roles of velocity anisotropy or rotation

+ **Galactic centers**

  e.g. (resonant) relaxation of eccentricity&orientation

+ **Dark matter halos**

  e.g. cusp-core transformation and environment